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APPLYING METHOD OF LINE ON CANTILEVER BEAM UNDER VARYING LOADS

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Abstract: The Method of Lines (MoL) is a powerful numerical technique for solving partial differential equations (PDEs) by discretising the spatial domain while keeping time as a continuous variable. In this study, we employ MoL to solve the fourth-order beam equation, which governs the dynamics of an Euler-Bernoulli cantilever beam. The primary objective is to develop a simplified computational model for a wind turbine blade, treated as a cantilever beam subjected to dynamic loading. The spatial discretisation is performed using finite difference approximations, transforming the governing PDE into a system of second-order ordinary differential equations (ODEs). The resulting ODE system is then integrated over time using Runge-Kutta.

Keywords: *Method of Lines, cantilever beam, varying loads, Euler-Bernoulli beam theory, simplified wind turbine blade*

1. INTRODUCTION

Wind energy plays a critical role in our transition to sustainable power, but it comes with challenges. One of the biggest being the impact of various forces on wind turbine blades. These blades experience vibrations and deflections due to aerodynamic forces, gravity, centrifugal effects, and even ice buildup in cold conditions. To keep turbines running efficiently and safely, we need accurate models that predict how blades move and respond under these forces.
One common way to simplify the study of wind turbine blades is by using Euler-Bernoulli beam theory, which helps describe how beams bend under force. In this study, we consider the blade as a structure that is fixed at the base and free at the tip. The mathematical equation governing its deflection is a complex fourth-order partial

differential equation (PDE), which is difficult to solve directly, especially for real-world conditions.

To tackle this, we use the Method of Lines (MoL), a numerical approach that converts the PDE into a system of ordinary differential equations (ODEs) by approximating spatial derivatives. Once converted, we use MATLAB's ode45 solver to compute the blade's response over time. This approach allows us to better understand and predict blade behaviour, contributing to the ongoing improvement of wind turbine performance.

2. Literary review

Many numerical approaches are used to analysis different aspects of wind turbine aerodynamics and vibrations discussed by (Al-Najjar & Jálics, 2024). One of these methods is the Method of Lines (MoL) and the Finite Difference Method (FDM). The FDM is used to discretise the governing partial differential equations and reduce it into a single domain which is usually the time domain. MoL is a useful numerical tool in many fields such as the Heat Conduction equation (Campo, 2022), Electromagnetic Waves equation (Pregla, 2008), in Fluid Dynamics (Zeng, 2024), and in Financial Mathematics (Haentjens, 2013).

Euler Bernoulli beam theory or Timoshenko beam theory are the main theories in beam structural dynamics. The MoL is used in cantilever beams analysis particularly for large deflection problems and requires iterative solutions (Khosravi & Jani, 2018), (Shahriar, Khodadadi, Majlesi, & Montoya, 2024). The discretisation in MoL transforms PDEs into ODEs by discretising the spatial variables to reduce the system to the time domain. This is achieved by approximating the spatial derivatives using finite differences or other discretisation methods (Pregla, 2008), (Leszczyński & Matusik, 2016), (Schiesser & Griffiths, 2009).

While it is possible to use the finite differences method to fully discretise the system of PDEs, the MoL is more efficient since other well-established methods like Runge-Kutta methods can be used to integrate the equations over the time domain (Campo, 2022), (Cash, 2005), (Yaseen & Batool, 2024).

MoL is also flexible and can be applied to a variety of fields and types of PDEs, including those in dynamic and static problems (Schiesser & Griffiths, 2009). It is particularly useful for problems involving wave propagation, heat conduction, and electromagnetic fields (Campo, 2022), (Schiesser & Griffiths, 2009).

3. MECHANICAL AND NUMERICAL MODELS

This numerical analysis used the Euler-Bernoulli beam theory under varying conditions (Figure 1). Euler-Bernoulli beam theory produced a fourth order partial differential equation for a simplified elastic beam under varying forces. It is used to calculate the small deflection of beams subjected to lateral loads which makes it a simplified Timoshenko–Ehrenfest beam theory (Szeidl & Kiss, 2020) shown in Equation 1.

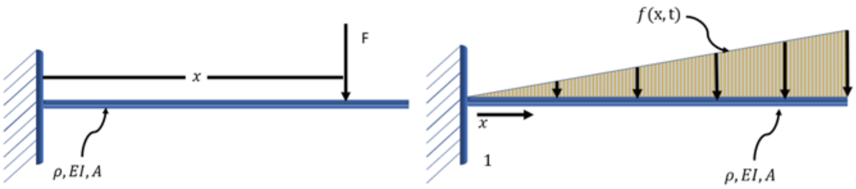

Figure 1. *Demonstrate Euler-Bernoulli beam theory*

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 u(x,t)}{\partial x^2}\right) = f(x,t) \tag{1}$$

where ρ is the density, E is the Young's modulus, I is the moment of inertia. These mechanical properties are set to 1 for the sake of simplicity. The term $u(x,t)$ is the solution of the PDE and it represents the vertical displacement at point x and time t. The term $f(x,t)$ is the force applied on the beam. The term $\frac{\partial^2 u(x,t)}{\partial x^2}$ is the curvature of the beam while $\rho \frac{\partial^2 u(x,t)}{\partial t^2}$ is the acceleration of the beam transverse vibration. The initial conditions for the problem are set as $u(x,0)$ and $\frac{\partial u(x,0)}{\partial t}$.

Figure 2. *Shows discretising of the space domain*

Figure 2 used to establish the space domain discretisation where the beam is divided into n number of segments, $n+1$ is number of nodes, and h is the length of each segment which equals the length of the beam l divided by n. The corresponding formula is shown as Equation 2 and 3

$$\rho \frac{\partial^2 u_j}{\partial t^2} + \frac{EI}{{h_j}^4}\left(u_{j-2} - 4\,u_{j-1} + 6\,u_j - 4\,u_{j+1} + u_{j+2}\right) = f_j \qquad (2)$$

where $0 < j < n-2$

$$\rho \begin{bmatrix} \frac{\partial^2 u_0}{\partial t^2} \\ \frac{\partial^2 u_2}{\partial t^2} \\ \vdots \\ \vdots \\ \frac{\partial^2 u_n}{\partial t^2} \end{bmatrix} + \frac{EI}{h^4} \begin{bmatrix} u_0 - 4\,u_1 + 6\,u_2 - 4\,u_3 + u_4 \\ u_1 - 4\,u_2 + 6\,u_3 - 4\,u_4 + u_5 \\ \vdots \\ \vdots \\ u_{n-4} - 4\,u_{n-3} + 6\,u_{n-2} - 4\,u_{n-1} + u_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ f_n \end{bmatrix} \qquad (3)$$

Boundary conditions at the fixed end using forward finite difference method are the displacement

$$u(0,t) = u_0(t) = 0, \qquad (4)$$

the slope (second order Forward FD)

$$\frac{\partial u(0,t)}{\partial t} = \frac{1}{2h}\left(-3u_0(t) + 4\,u_1(t) - u_2(t)\right) = 0, \qquad (5)$$

while the Boundary conditions at the free end $x = l$ are the Bending moment (second order Backward FD)

$$EI\frac{\partial^2 u(l,t)}{\partial t^2} = \frac{EI}{h^2}\left(-u_{n-3}(t) + 4\,u_{n-2}(t) - 5u_{n-1}(t) + 2u_n(t)\right), = 0 \qquad (6)$$

where l the length of the blade. The transverse Shear force (second order Backward FD) is calculated using the following equation:

$$EI\frac{\partial^3 u(l,t)}{\partial t^3} = \frac{EI}{2h^3}(3u_{n-4}(t) - 14\,u_{n-3}(t) + 24u_{n-2}(t) - 18u_{n-1}(t) + 5u_n) = 0. \quad (7)$$

These equations are then implemented in MATLAB where a code was developed to solve them under the desired conditions by using ode45 for the desired time.

4. RESULTS

Wind turbines may experience pre-bending conditions due to ice accumulating over the blade. When the ice is removed due to wind, gravity or the melting of ice, it can alter the structural state of the blade. In this case, this is modelled as initial conditions, for example, $u(x, t = 0) = g(x)$ where $g(x)$ is a function that describes the pre-bending state of the beam.

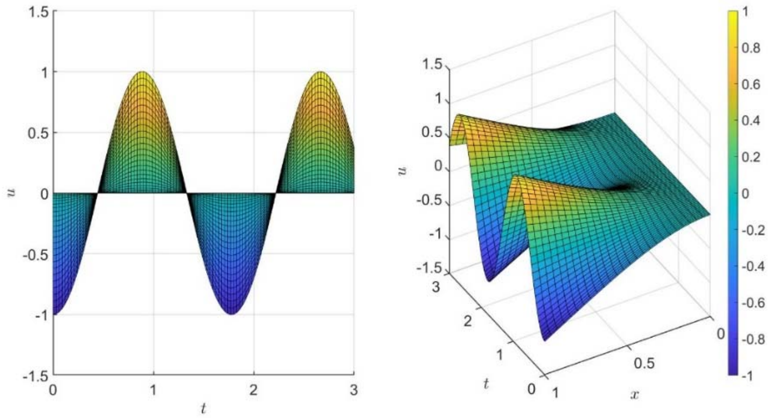

Figure 3. *Free vibration of a cantilever beam with initial condition $u(x, 0)$ as the first mode shape for a span of 3 sec*

Figure 3Figure 4, andFigure **5** illustrate the free vibration of a beam for three seconds. The initial conditions are the first mode shape, the second mode shape under free vibration and $u(x, 0) = x^4$, respectively. The x-axis represents the length of the beam where the beam is fixed at $x = 0$ and free at the other end. The t-axis represents the time that spans for three seconds, which is enough time for the beam to oscillate at least once. The u-axis represents the displacement (deformation) of the beam. Since this is free vibration the beam maximum and minimum displacement values depends on the initial conditions, the length of the beam, and the beam mechanical properties.

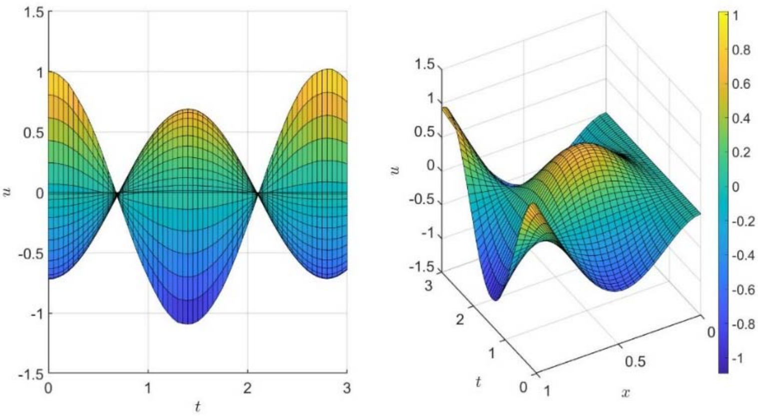

Figure 4. *Free vibration of a cantilever beam with initial condition* $u(x, 0)$ *as the second mode shape for a span of 3 sec*

The solution for the first two mode shapes is from chapter 11 of (Rao, 2006). First mode shape is described in the following equation:

$$u_1 = 0.5\cos(1.875\,x) - 0.5\cosh(1.875\,x) - 0.367\sin(1.875\,x) + 0.367\sinh(1.875\,x). \quad (8)$$

Second mode shape is described in the next equation:

$$u_2 = 0.5\cos(4.69\,x) - 0.5\cosh(4.69\,x) - 0.509\sin(4.69\,x) + 0.509\sinh(4.69\,x). \quad (9)$$

In Figure 6, the forced vibration is modelled by changing the B.C at the free end to be equal to a mass $\partial^3 u(l,t)/\partial t^3 = 1$ where the mass is equal $\mathrm{m} = \frac{1}{\mathrm{g}}\ kg$ when g is gravitational acceleration constant while Figure 7 and Figure 8 illustrate the forced vibration of cantilever beam for $f = 10\ sin\ (10\ tx)$ and $f = 5\ cos(10\ tx)$, respectively. The oscillating force at the free end gives insight to the behaviour of the beam under simplified ice slip conditions while the attached mass is for a simplified ice accumulating on the beam.

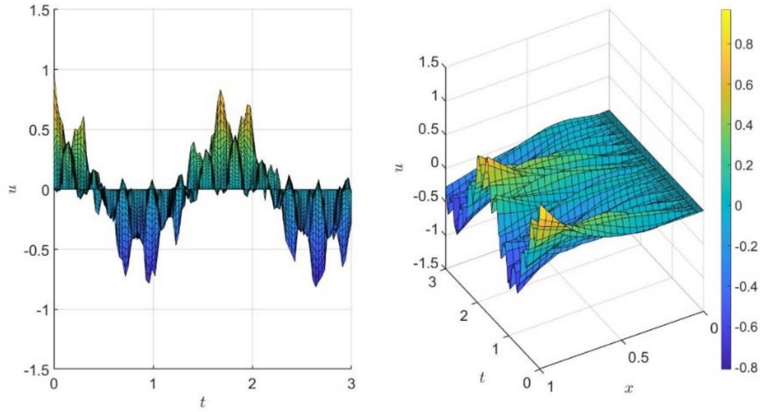

Figure 5. *Free vibration of a cantilever beam with initial condition $u(x,0) = x^4$ for a span of 3 sec*

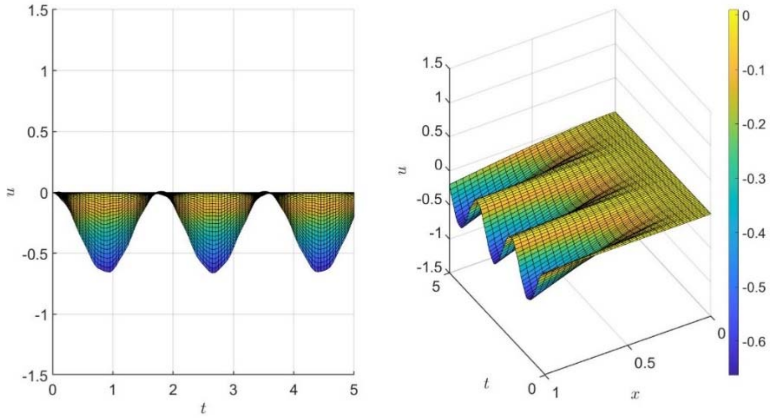

Figure 6. *Free vibration of a cantilever beam a mass attached at its free end $m = \frac{1}{g}$ for a span of 5 sec*

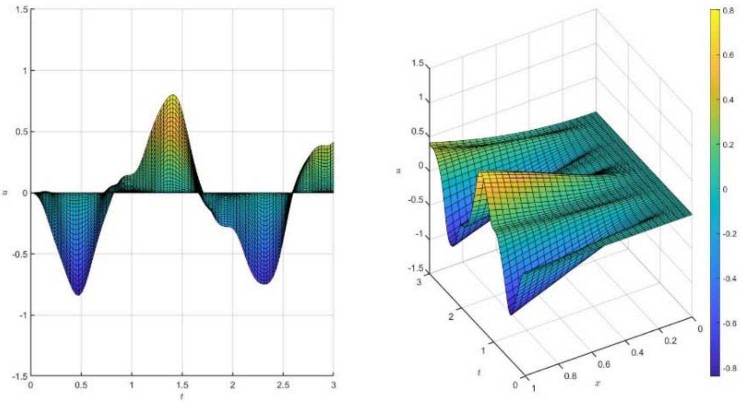

Figure 7. *Free vibration of a cantilever beam with a force on its free end* $\partial^3 u(l,t)/\partial t^3 = 10\ sin\ (10\ tx)$ *for a span of 3 sec*

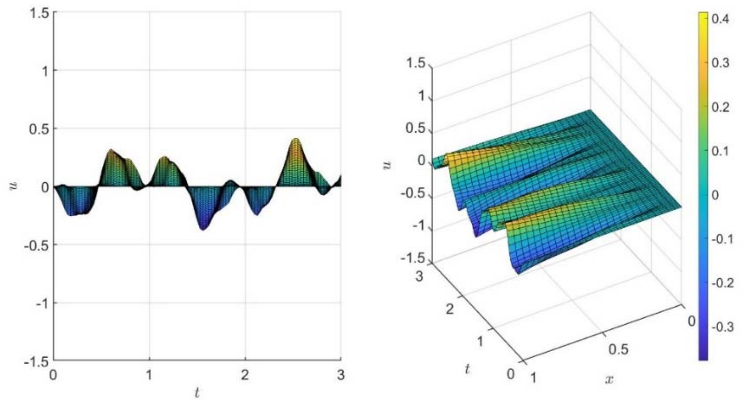

Figure 8. *Free vibration of a cantilever beam with a force on its free end* $\partial^3 u(l,t)/\partial t^3 = 5\ cos\ (10\ tx)$ *for a span of 3 sec*

5. SUMMARY

Euler-Bernoulli beam theory was used to describe the vibration of a simplified wind turbine model with fixed support at the root and free end at the tip. After that, finite difference method was used to discretise the space domain and create a system of ODEs. Then the system was solved using ode45 in MATLAB. The free and forced vibration of the beam was solved under different conditions. In case of free vibration, a pre-bended beam was modelled by changing the I.Cs while different forces were attached at the free end to see the forced vibration.

Since Euler-Bernoulli beam theory does not consider the damping effect or any type of energy dissipation, the results will only keep oscillating in the same pattern. The absolute values of the minimum and maximum displacement depend on the applied force proportionally, while the increase in the beam rigidity EI decreases the absolute values of the minimum and maximum peaks. Finally, the results demonstrated the effect of initial pre-bending conditions and external forces on the dynamic response of the beam, that can be validated in the future using modal analysis experiment (Al-Najjar, Jálics, & Kollár, 2023), (Al-Najjar, Kollár, & Jálics, 2024). Euler-Bernoulli beam theory may give valuable insights regarding the beam vibration and oscillation, its limitation lies in neglecting energy dissipation effects make its prediction different from real world vibration, especially in case of elastic and long wind turbine blade. Future work could incorporate the damping effects of the beam improving the accuracy of the model.

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