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DEVELOPMENT OF CALCULATING PROCESS FOR ELASTO-HYDRODYNAMIC SPOT CONTACT BY P-FEM

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Rolling-sliding machines such as gears, cams and followers, and bearings, which are often subjected to high loads, high speeds and high slip conditions when not only the pressure distribution in the lubricant and variation of the viscosity due to pressure is a question but the surface deformation to calculate the sub-surface stress field. To analyse the tribological conditions of the contact surfaces, after Osborne Reynolds (1886) [1], who presented the theory of hydrodynamic lubrication and the governing partial differential equation for the pressure field, the generalized Reynolds equation was developed by Dowson (1962) [7]. Although several methods have been already developed for solving EHD problems, the solution of the highly nonlinear problem is still quite challenging. Unfortunately, only few solutions were published based on FEM despite it permits the using of polynomial approximation for powerful calculation of the generalized Reynolds equation without smooth mesh. However, applying rough mesh, the end of the contact can locate far from the boundaries of the elements. This fact makes to be important to manage the cavitation inside elements. So the development of a p-version FEM model for calculating the film shape, the pressure and temperature distribution and its implementation to commercial software seems to be timely. For this reason, special lubricant film element can be developed for such problems where pressure and film thickness can be handled as independent element variables in the generalized Reynolds equation and determined incrementally by fully coupling the hydrodynamic and the solid mechanic problem.

1. Basic equations and boundary conditions for the description of the EHD phenomenon of lubrication theory

The generalized case of surface pairs contacting along a spot in the status of liquid friction is illustrated in Figure 1.The gap between the bodies is filled with lubricant due to the relative motion of the bodies and hydrodynamic pressure develops due to the movement of the lubricant. The movement of the lubricant is caused by the tangential (shear) stress generated in the lubricant as the result of the relative motion of the surfaces. At the particular kinematic condition of the contacting bodies and with a given gap geometry the pressure distribution acting on the surfaces is able to maintain balance with the force pressing the surfaces to each other and prevent a direct body-to-body contact thereby. It can be seen, therefore, that if the circumstances of contact developing under thermal elasto-hydrodynamic conditions of lubrication are wished to be modelled then hydrodynamic, thermodynamic and strength problems – forming highly non-linear systems even by themselves but also because of the status-dependence of the properties of the various continuums – have to be solved at the same time and in combination.

Lubrication analysis began looking into the finite element formulation in the sixties by Reddi (1969) [2], the seventies by Rohde and Oh (1975) [3], and they found the method promising and reliable. Freund and Tieu (1993) [4] used h-FEM for solving the generalized

Reynolds equation. In these investigations, the traditional h-version finite element method (h-FEM) had been used in the formulation of lubrication problems till 1991 when Nguyen [5] reported an analysis with using p-version finite element method but the geometry and the oil properties were assumed to be constant. However the p-approximation concept was introduced by Babuska and Szabó. (1979) [6], but most p-version formulation appear in the literature deal with problems in solid mechanics and heat transfers.

In this work the application possibility of p-FEM for EHD problem is presented with the necessary developments. For FEM governing equation the weak form of the weighted-residual integral form of the Reynolds has been applied. Lagrange functions have been used for the polynomial approximation of the unknown pressure distribution. The p-FEM based solution of EHD problems; the polynomial approximation makes the opportunity of replacement of the smooth mesh with a rough one. In order to have the compatible formulation to the well-known FEM solutions, the film shape has been calculated as a superposition of the discretized original geometry, the displacement of a rigid surface and the discretized deformation of the surface under pressure.

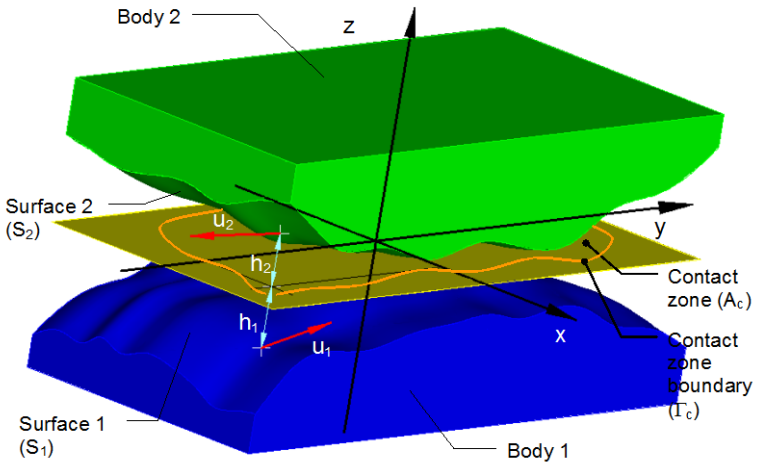


Figure 1: Contacting bodies

2. Hydrodynamic problem

For calculating the contact pressure due to fluid film lubrication the generalized Reynolds equation was developed by Dowson (1961) [7] as a partial differential equation take account the changes the viscosity across the film thickness. Based on the concepts of it, the weak integral form of the generalized Reynolds equation is:

$$\int_{A_C} \nabla_{x,y} w \cdot \Phi \cdot \nabla_{x,y} p \cdot dA - \int_{A_C} \nabla_{x,y} w \cdot \Psi \cdot dA - \int_{A_C} w \cdot \Omega \cdot dA + \oint_{\Gamma_C} w \cdot q_S \cdot ds = 0 \qquad (1)$$

where:

w: a weight function,

q: lubricant in or outlet.

$$\Phi = \mathrm{F}_2 + \mathrm{G}_1 \tag{2}$$

$$\begin{aligned}\boldsymbol{\Psi} = {} & h_2 \cdot \rho\big|_{z=h_2} \cdot \mathbf{u_{xy}}\Big|_{z=h_2} + h_1 \cdot \rho\big|_{z=h_1} \cdot \mathbf{u_{xy}}\Big|_{z=-h_1} - \\ & - \frac{F_3 + G_2}{F_0} \cdot \left(\mathbf{u_{xy}}\Big|_{z=h_2} - \mathbf{u_{xy}}\Big|_{z=-h_1} - \mathbf{K}_{0xy} \right) - G_3 \cdot \mathbf{u_{xy}}\Big|_{z=-h_1} - \mathbf{K}_{1xy} - \mathbf{K}_{2xy}\end{aligned} \tag{3}$$

$$\Omega = \rho\big|_{z=h_2} \cdot \mathbf{u_{xy}}\Big|_{z=h_2} \cdot \nabla_{x,y} h_2 + \rho\big|_{z=-h_1} \cdot u_z\big|_{z=-h_1} - \rho\big|_{z=h_2} \cdot u_z\big|_{z=h_2} - \int_{-h_1}^{h_2} \frac{\partial \rho}{\partial t} \cdot dz \tag{4}$$

Employing the customary decomposition:

$$u_z\big|_{z=-h_1} = -\mathbf{u_{xy}}\Big|_{z=-h_1} \left(\nabla_{xy} h_1 \right) + W_1(t,x,y) \tag{5}$$

$$u_z\big|_{z=h_2} = \mathbf{u_{xy}}\Big|_{z=h_2} \left(\nabla_{xy} h_2 \right) + W_2(t,x,y)$$

for the components in gap thickness direction of surface velocities Ω will take the following form:

$$\Omega = \rho\big|_{z=-h_1} \cdot W_1 - \rho\big|_{z=h_2} \cdot W_2 - \int_{-h_1}^{h_2} \frac{\partial \rho}{\partial t} \cdot dz \tag{6}$$

$\mathbf{u}_{\mathrm{xy}}{}^{\mathrm{T}} = [u_x, u_y]$: the velocity in the contact plane, u_z: the velocity in the z direction,

$\nabla_{\mathrm{x,y}} = [\partial/\partial \mathrm{x}, \partial/\partial \mathrm{y}]$

F_i,G_i, $\mathbf{K}_{\mathrm{i_{xy}}}$: viscosity-density functions:

$$F_0 = \int_{-h_1}^{h_2} \frac{F(\tau_{eq})}{\tau_{eq}} dz \qquad F_1 = \int_{-h_1}^{h_2} \frac{F(\tau_{eq})}{\tau_{eq}} z dz \qquad F_2 = \int_{-h_1}^{h_2} \rho \frac{F(\tau_e)}{\tau_e} z^2 dz - \frac{F_1 F_3}{F_0}; \quad F_3 = \int_{-h_1}^{h_2} \rho \frac{F(\tau_e)}{\tau_e} z dz$$

$$G_1 = \int_{-h_1}^{h_2} z \frac{\partial \rho}{\partial z} \left(\int_{-h_1}^{z} \frac{F(\tau_e)}{\tau_e} \hat{z} d\hat{z} - \frac{F_1}{F_0} \int_{-h_1}^{z} \frac{F(\tau_e)}{\tau_e} d\hat{z} \right) dz; \qquad G_2 = \int_{-h_1}^{h_2} z \frac{\partial \rho}{\partial z} \left(\int_{-h_1}^{z} \frac{F(\tau_e)}{\tau_e} d\hat{z} \right) dz; \; G_3 = \int_{-h_1}^{h_2} z \frac{\partial \rho}{\partial z} dz$$

$$\mathbf{K}_{0xy} = \int_{-h_1}^{h_2} A \frac{d\bar{\sigma}'_z}{dt} dz = \begin{bmatrix} \int_{-h_1}^{h_2} \frac{A d\tau_{xz}}{dt} dz \\ \int_{-h_1}^{h_2} \frac{A d\tau_{yz}}{dt} dz \end{bmatrix} \quad \mathbf{K}_{1xy} = \int_{-h_1}^{h_2} z \rho A \frac{d\bar{\sigma}'_z}{dt} dz \quad \mathbf{K}_{2xy} = \int_{-h_1}^{h_2} z \frac{\partial \rho}{\partial z} \left(\int_{-h_1}^{z} A \frac{d\bar{\sigma}'_z}{dt} d\hat{z} \right) dz$$

In case of steady-state rolling-sliding contacts $\Omega = 0$ and $\mathbf{K}_{\mathrm{i_{xy}}} = 0$ furthermore G_i negligible.

For simple model, the film shape can be calculated as a superposition of the original geometry, the displacement of a rigid surface and the deformation of a half-space under pressure.

After deformation, the film shape:

$$h = h_1 + h_2 = h_{g_1} + \Delta_{rigid\,1} + \delta_{1_1} + h_{g\,2} + \Delta_{rigid\,2} + \delta_2 = h_g + \Delta_{rigid} + \delta \tag{7}$$

where Δ_{rigid} is the displacement as a rigid surface, h_g is the undeformed gap size, the δ deformation of the surfaces. Usually it can be assumed that $h_1=0$, $h_2=h$.

The integral of the pressure over the contact area should be equal with the load.

$$F_W = \int_{A_c} p \cdot dA \tag{8}$$

F_W is the normal load of the surfaces. Loadcase can be satisfied if the Δ_{rigid} is a variable.

3. Finite Element Formulation

For the discretized governing equations, the weak form of the weighted-residual integral form of the Reynolds equation has been applied. Lagrange functions [6] have been used for the polynomial approximation of the un-known pressure and surface deformation.

In the case of variation methods, the integral forms of the differential equations are used and in the course of this, rationally, certain quantities have to be integrated over the region investigated. The integration range has to be divided into shapes characteristic of a particular element type for the use of the finite element method and then derived into a unified shape by means of conform transformation for numerical integration.

The edges and sides of the elements along the gap are parallel with coordinate axis z and thus the unit vector $\boldsymbol{e}_z$ of the global coordinate system connected to the contact surface and unit vector $\boldsymbol{e}_\zeta$ of the coordinate system connected to the element coincide. Consequently x and y coordinates, the gap size and the pressure are not the functions of ζ:

$$x^e(\xi,\eta,t) = \sum\nolimits_i X_i^e(t) N_{x\,i}^e(\xi,\eta) = \mathbf{N}_x^{eT}(\xi,\eta)\mathbf{X}^e(t) \tag{9}$$

$$y^e(\xi,\eta,t) = \sum\nolimits_j Y_j^e(t) N_{y\,j}^e(\xi,\eta) = \mathbf{N}_y^{eT}(\xi,\eta)\mathbf{Y}^e(t) \tag{10}$$

$$h_{g_i}^e(\xi,\eta,t) = \sum\nolimits_j H_{g_i\,j}^e(t) N_{h\,j}^e(\xi,\eta) = \mathbf{N}_h^{eT}(\xi,\eta)\mathbf{H}_{g_i}^e(t) \tag{11}$$

$$\delta_i^e(\xi,\eta,t) = \sum\nolimits_j H_{\delta_i\,j}^e(t) N_{h\,j}^e(\xi,\eta) = \mathbf{N}_h^{eT}(\xi,\eta)\mathbf{H}_{\delta_i}^e(t) \tag{12}$$

$$h_i^e = \left[\mathbf{N}_h^{eT}, 1\right] \begin{bmatrix} \mathbf{H}_{g\,i}^e + \mathbf{H}_{\delta i}^e \\ \Delta_{rigid\,i} \end{bmatrix} = \left[\mathbf{N}_h^{eT}, 1\right] \begin{bmatrix} \mathbf{H}_i^e \\ \Delta_{rigid\,i} \end{bmatrix} \tag{13}$$

$$p^e(\xi,\eta,t) = \sum\nolimits_j P_j^e(t) N_{p\,j}^e(\xi,\eta) = \mathbf{N}_p^{eT}(\xi,\eta)\mathbf{P}^e(t) \tag{14}$$

The approximation of coordinate z is necessary as well because of the thermodynamical problem. Since z is linear in ζ:

$$z^e = -h_1^e\left(\frac{1-\zeta}{2}\right) + h_2^e\left(\frac{1+\zeta}{2}\right) = -\left(\frac{1-\zeta}{2}\right)\mathbf{N}_h^{eT}\mathbf{H}_1^e + \left(\frac{1+\zeta}{2}\right)\mathbf{N}_h^{eT}\mathbf{H}_2^e \tag{15}$$

Since the Rayleigh-Ritz finite element model was developed, $w = \mathbf{N}_p^e$ and the following nonlinear equation system was given per elements:

$$\begin{aligned}&\mathbf{K}^{\mathbf{e}}\left(\eta(p,\vartheta),\rho(p,\vartheta),h(p)\right)\cdot\mathbf{P}^e = \\ &= \mathbf{f}_a^e\left(\eta(p,\vartheta),\rho(p,\vartheta),h(p)\right) + \mathbf{f}_b^e\left(\eta(p,\vartheta),\rho(p,\vartheta),h(p)\right) - \mathbf{Q}^e\end{aligned} \tag{16}$$

where:

$$\mathbf{K}^e = \int_{A^e}\Phi^e\cdot\left[\mathbf{B}_p^e\circ\mathbf{B}_p^e\right]\cdot dA^e\ ,\ \mathbf{B}_p^e = \nabla_{xy}\mathbf{N}_p^{eT} \tag{17}$$

$$\mathbf{f}_a^e = \int_{A^e}\partial\mathbf{N}_p^e\cdot\mathbf{\Psi}^e\cdot dA^e\ ,\mathbf{f}_b^e = \int_{A^e}\mathbf{N}_p^e\cdot\Omega^e\cdot dA^e\ ,\ \tilde{\mathbf{Q}}^e = \oint_{\Gamma^e}q_h^{n_\Gamma^e}\mathbf{N}_p^e d\Gamma \tag{18}$$

4. Penalty cavitation algorithm

For p-FEM, it is important to manage the cavitation inside elements instead of deactivating the elements where the cavitation occurs.

The connection between the viscosity and pressure has been applied according to Kumar and Booker [8][9] for further investigation as the equation (19) shows.

$$\frac{\eta}{\eta_L} = \frac{\rho}{\rho_L} \qquad \rho \le \rho_c \tag{19}$$

However the density change as a function of the pressure in the cavitation zone let be approximated by a high gradient slope under the cavitation pressure. The difference from the original Kumar and Booker model can be seen in the Figure 2. The gradient of the slope should depend on a parameter that can be called penalty parameter. The higher value of the penalty parameter is the higher gradient of the density in the cavitation zone. It should be taken into consideration as well that the density should be higher than zero everywhere.

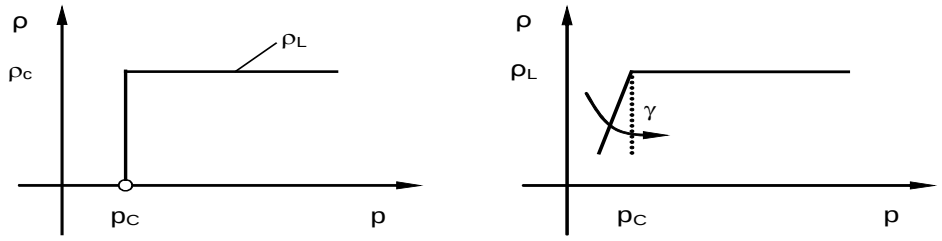


Figure 2: Penalty parameter approximation of the density

The above described criteria can be satisfied with the following density function:

$$\rho^{*} = \frac{\rho_L(p,\vartheta)}{\gamma(p)\cdot(p_c - p)+1} \tag{20}$$

where γ(p) is the penalty function which is:

$$\gamma(p) = \begin{cases} 0 & if \quad p > p_c \\ c & if \quad p \le p_c \end{cases} \tag{21}$$

The ρ* is valid in the lubrication region and the cavitation zone as well, and the volume fraction can be written as:

$$\theta(p) = \frac{\rho^{*}}{\rho_L} = \frac{1}{\gamma(p)(p_c - p)+1} \tag{22}$$

Applying the linear correlation between the density and the viscosity according to Kumar and Booker the viscosity can be written as follows:

$$\eta^{*} = \eta_L(p,\vartheta)\frac{\rho^{*}}{\rho_L(p,\vartheta)} = \theta(p)\eta_L(p,\vartheta) \tag{23}$$

All the parameters needed for the EHD analysis can be written as a function of the pressure in the cavitation zone, too.

5. Linearization of the discretized Reynolds equation

The difficulty of reaching a solution in the subject matter investigated is caused primarily by the strongly nonlinear nature of the Reynolds equation. The solutions of strongly nonlinear equations are based in the vast majority of cases on the Newtonian or gradient methods.

Consequently, the discretized Reynolds equation system has to be linearized in order to enable the determination of the pressure distribution and the associated gap size. Let us observe the equation as:

$$\mathbf{R} = \mathbf{R}\left(\mathbf{P}, \rho, \dot{\rho}, h, \vec{r}, t, \eta, \tau_{eq}, \vartheta, \theta, \vec{u}_1, \vec{u}_2 .....\right) \tag{24}$$

The linearized form of this at an arbitrary point $\mathbf{P}=\mathbf{P}^{j}$:

$${}^{i}\mathbf{R}\left(\mathbf{P}^{j}, {}^{i-1}\mathbf{R}\right) + \left(\frac{\partial^{i}\mathbf{R}}{\partial \mathbf{P}} + \frac{\partial^{i}\mathbf{R}}{\partial \rho}\frac{\partial \rho}{\partial p}\mathbf{N}_p + \frac{\partial^{i}\mathbf{R}}{\partial \eta}\frac{\partial \eta}{\partial p}\mathbf{N}_p + \frac{\partial^{i}\mathbf{R}}{\partial h}\mathbf{N}_h\mathbf{D}_p + ... \right)\Bigg|_{\mathbf{P}=\mathbf{P}^{j}} \Delta\mathbf{P} + O = 0 \tag{25}$$

The solution of ${}^{i}\mathbf{P}^{*}$ at residual ${}^{i}\mathbf{R} = 0$ is approximated with a series of solutions for $\Delta\mathbf{P}^{j}$ in this equation as shown below:

$$\mathbf{P}^{j+1} = \mathbf{P}^{j} + \alpha \cdot \Delta\mathbf{P}^{j} \qquad \alpha = [0..1] \tag{26}$$

As the initial system of equations is highly nonlinear in the case of EHD problems, it is very difficult to find an initial state where the instability of the solution can be avoided. For this reason, a properly chosen α is required. For this reason, it is expedient to determine the optimum damping rate in such a way that it should result in the largest possible reduction of the residual value. This can be ensured, obviously, only by some properly chosen methods.

6. Implementation to FEM software

The Comsol Multiphysics software was used for calculating EHD problems. The Comsol Multiphysics use the finite element formulation with Lagrange test functions as it was shown above to solve numerical problems.

It has got modules which are useful to build up an EHD model. The weak formulation of the hydrodynamic problem described by (1) equation can be created in the Mathematics module and model can be freely modified. The elastic deformation of the surface and the rigid displacement of the surfaces can be calculated in the Structural Mechanics module which uses the common structural finite element methods. The Comsol Multiphysics also provides an opportunity to incorporate function of penalty cavitation described by (20)–(23) equations.

The EHD as it was mentioned above is a highly nonlinear problem. The Comsol Multiphysics use the damping Newton-Raphson method described by (24)–(26) equation for the nonlinear problems. The method of determining the damping rate is the following:

- Firstly, the method calculate an **E** relative error value with the (27) equation

$$f'(\mathbf{P}^j)\mathbf{E} = -f(\mathbf{P}^{j+1}) \tag{27}$$

- If the relative error value is higher or equal to the **E** obtained from previous iteration step, then the value of the damping rate is reduced
- This process is repeated until the relative error value is lower than the value obtained from the previous iteration step or the minimum attenuation rate is achieved

After the determining the damping rate, the Newton-Raphson method calculate the value of the $\Delta\mathbf{P}^{j+1}$. The iteration continues until the special error norms do not achieve the value of relative tolerance.

The relative tolerance, the maximal iteration step number and the minimum damping rate can be manually given. The smaller the value of relative tolerance, the more accurate result but we risk that the iteration does not converge due to numerical errors.

7. Solved cases

The next three cases were evaluated with our calculating method introduced above:

Hydrodynamic lubrication between two plain surfaces:

- The geometrical parameters of the gap: $h_0 = 10$ μm; $h_1/h_0 = 1.25$; $l = 25$ mm as can be seen on Figure 3. Width of the plain surfaces:100 mm
- The difference of the velocity of the plains: $U = u_x = 25$ m/s; $V = u_y = 0$ m/s
- At the edges of the gap, the pressure is 0 MPa
- Viscosity of the lubricant: $\eta = 5 \cdot 10^{-2}$ Pa·s

- The upper surface is rigid and the lower surface is linear elastic: E = 109.9 GPa
- External load: W_{ext} = 0.1 MN

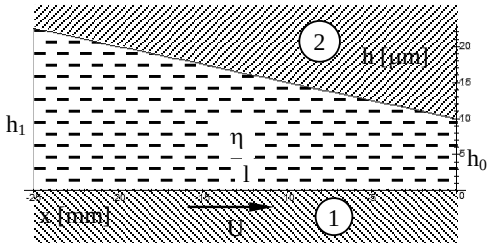


Figure 3: Initial gap geometry between two plain surfaces

Results:

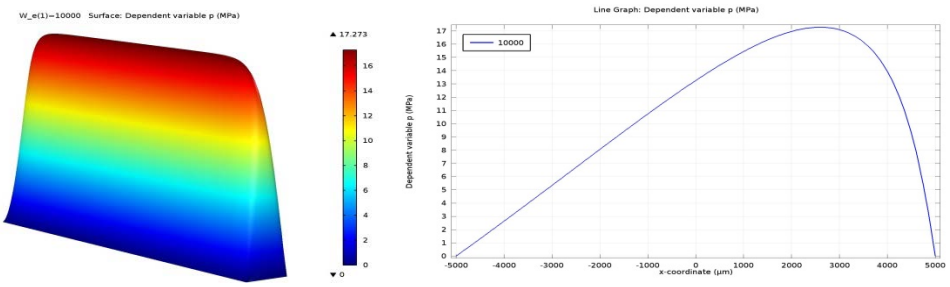


Figure 4: The pressure distribution in the gap between two plain surfaces and in the middle cross-section

Hydrodynamic lubrication between double parabolic and plain surface taking into account the cavitation

- Length of a gap: l = 76.2 mm as can be seen on Figure 5.
- Length of a bearing: l_b = 38.1 mm
- Minimal gap height: h_0 = 25,4 μm
- Maximal gap height: h_1 = 50,8 μm
- Velocity to the x direction of the bearings: U = 4,57 m/s
- Viscosity of the lubricant: η = 0,039 Pa·s
- The penalty parameter of the cavitation: $c = 1{,}9 \cdot 10^{-5}$.

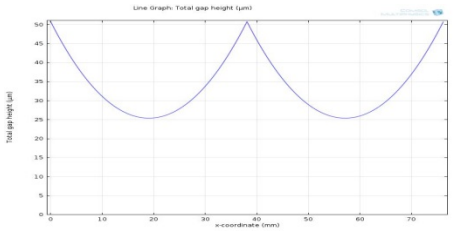


Figure 5: Total gap height between double parabolic and plain surface

Results:

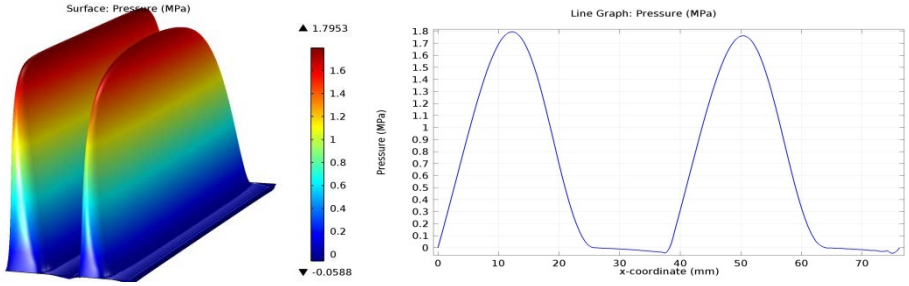


Figure 6: The pressure distribution in the gap between double parabolic and plain surface and in the middle cross-section

Elastohydrodynamic lubrication of roller-bearing

- The reduced geometrical parameters of the gap:
  - initial minimum gap height: $h_0 = 0.228127$ µm
  - radius of the roller-bearing: r = 30.031035 mm
  - width of the plain surface of the roller: 30 mm

  Schematic figure of the gap can be seen on Figure 10.
- The difference of the velocity of the surfaces: $U = u_x = 1.58853$ m/s; $V = u_y = 0$ m/s
- At the edges of the gap, the pressure is 0 MPa
- The upper surface is rigid and the lower surface is linear elastic: E = 109.9 GPa
- Viscosity of the lubricant described by Barus model: $\eta = 1.539 \cdot 10^{-2} \cdot e^{\alpha p}$ Pa·s, where $\alpha = 5000 / E$
- External load: $W_{ext} = 500$ N
- The penalty function of the cavitation: $c = 1{,}9 \cdot 10^{-5}$

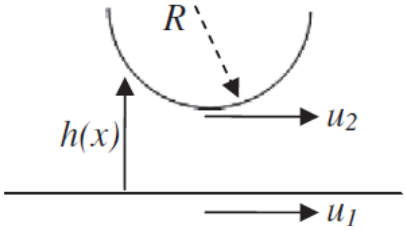


Figure 7: The reduced gap geometry of roller-bearing

Results:

This test case was solved in two steps. At first the problem was solved as a line contact. The result can be seen on the Figure 8: The pressure distribution in the gap of roller-bearing in case of line contact.. The results are in good agreement with the literature. Secondly it was solved as a finite length contact. Unfortunately the solution could not converge due to numerical instability at the side of the roller. The results of the last converged step can be seen on *Figure 10* and *Figure 12*.

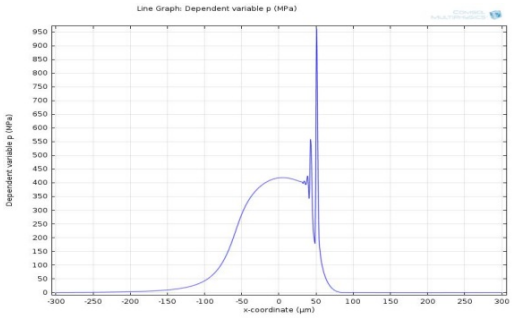


Figure 8: The pressure distribution in the gap of roller-bearing in case of line contact.

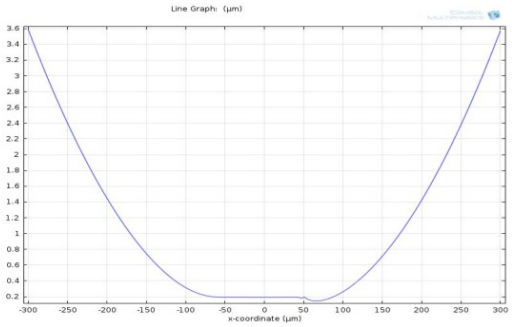


Figure 9: The gap size in the gap of roller-bearing in case of line contact

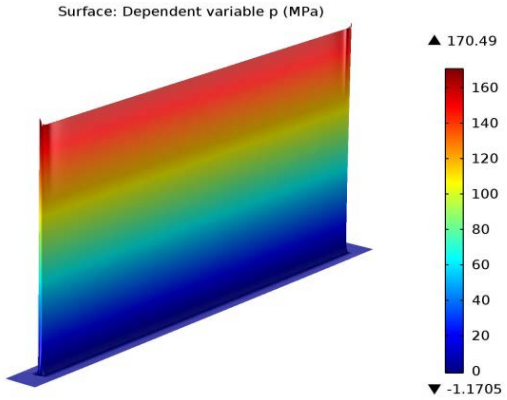


Figure 10: The pressure distribution in the gap of roller-bearing

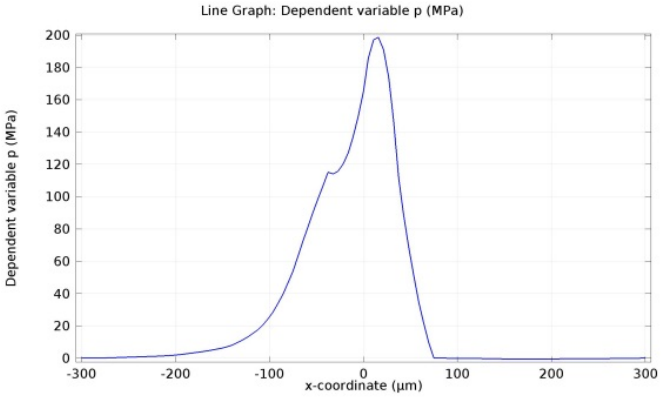


Figure 11: The pressure distribution in the middle cross-section of the gap of roller-bearing

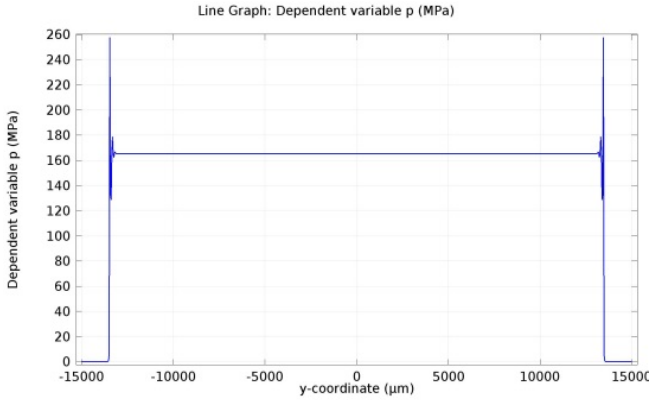


Figure 12: The pressure distribution along the length of roller

8. Conclusion

For the three-dimensional contact problem of lubrication, a two-dimensional lubrication fluid film finite element was developed and implemented to a commercial code. Presented penalty parameter method for viscosity is applicable for the advanced numerical methods, such as for the p-version finite element method. Summarising the results of the calculations we can state that the p-FEM method introduced above is a good and stable process to solve the EHD lubrication problems. It reduce the model size by decreasing the number of elements and it can be implemented for other problem types as well, such as rough surfaces, dynamically loaded bearings. However further effort is needed to handle the numerical instability in case of finite length contact.

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