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APPLICABILITY OF A CONSTANT YOUNG'S MODULUS IN GEOMETRICALLY NONLINEAR ELASTICITY

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Dedicated to Professor István Páczelt on the occasion of his sixtieth birthday

Abstract. The aim of this paper is to demonstrate through a simple problem that the use of a constant Young's modulus in numerical analyses of geometrically nonlinear elasticity problems should be considered as a potential source of inaccuracy, depending on what work-conjugate stress and strain measures the formulation uses. Importance of the Biot stresses and Jaumann strains as conjugate engineering stress and strain measures in nonlinear elasticity is emphasized.

Keywords: Stress and strain measures, constant Young's modulus, geometrically nonlinear elasticity

1. Introduction

In geometrically nonlinear elasticity problems the material is often considered to be linearly elastic and the nonlinearity usually enters into the formulation through the strain-displacement relations, due to the appearance of larger displacement derivatives (which are mostly related to the large local rotations). In the majority of numerical analyses, the preferred work-conjugate stress and strain measures are the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor.

When the stretches of the material curves of an elastic body are much smaller than unity, the constant Young's modulus measured for a given material can directly be applied between different work-conjugate stress and strain measures such as the popular Green-Lagrange strains and the second Piola-Kirchhoff stresses. For some modern materials, applied for instance in manufacturing advanced composite structures, the limit of the linearly elastic behavior in terms of stretches can, however, be much higher than that for classical materials (e.g. for metals). This means that in many geometrically nonlinear elasticity problems, the stretches in some parts of the body can be much larger than in other parts of the body and using the same constant Young's modulus for relating the second Piola-Kirchhoff stresses to the Green-Lagrange strains in each point of the body, independently of the local stretch value, can lead to incorrect numerical results [1].

In this paper we consider the uniaxial tension of a homogeneous isotropic prismatic beam in the elastic range. Although the material of the beam is assumed to be linearly elastic, it is not assumed that the stretch of the beam is much smaller than unity. The deformation and the stress state of the beam will be described by stress and strain measures of finite elasticity, which is an important issue for the purpose of the present investigations. In contrast to [1], the relationship between the Cauchy stresses and the Euler-Almansi strains is, however, not assumed to be linear. Instead, as in reality, the dependence of the nominal stress on the stretch of the beam is considered to be linear and the tangent of this linear function is the (constant) Young's modulus of the material of the beam.

After writing down the different work-conjugate stress and strain tensors for the beam under uniaxial tension, the relationships between the normal stresses and strains in the axial direction are derived and the error resulting from the use of a constant Young' modulus at different stretch and strain levels is investigated.

2. Strain measures

Consider the elastic deformation of a homogeneous isotropic prismatic beam of length L under uniaxial tension. Let A_0 and A be the cross-sectional areas of the beam in the reference (undeformed) and current (deformed) configurations, respectively. Let the two configurations of the beam be investigated in the same Cartesian frame. Coordinates of material points in the reference and current configurations are denoted by X, Y, Z and x, y, z, respectively, where Z and z are the axes of the beam in the two configurations. The axial force is denoted by F and the change in length L of the beam is denoted by $\triangle L$.

Plotting the nominal stress $\sigma = F/A_0$ against the stretch $\epsilon = \triangle L/L$ of the beam, the stress-strain curve is obtained. We restrict our investigations here to the elastic range of the deformation when the nominal stress σ is a linear function of the stretch ϵ. The elasticity or Young's modulus E_Y of the material of the beam is defined as the tangent of this linear function $\sigma = \sigma(\epsilon) = E_Y\,\epsilon$.

The deformation gradient and the inverse deformation gradient of the beam under uniaxial tension are given by

$$\mathbf{F} = \begin{bmatrix} 1-\nu\epsilon & 0 & 0 \\ 0 & 1-\nu\epsilon & 0 \\ 0 & 0 & 1+\epsilon \end{bmatrix}, \tag{2.1}$$

$$\mathbf{F}^{-1} = \begin{bmatrix} (1-\nu\epsilon)^{-1} & 0 & 0 \\ 0 & (1-\nu\epsilon)^{-1} & 0 \\ 0 & 0 & (1+\epsilon)^{-1} \end{bmatrix}, \tag{2.2}$$

where ν is the Poisson's ratio. Since the deformation of the beam is rotation-free, the polar decomposition of the deformation gradient reads

$$\mathbf{F} = \mathbf{R}\cdot\mathbf{U} = \mathbf{I}\cdot\mathbf{U} = \mathbf{U}\,, \tag{2.3}$$

where $\mathbf{U}$ is the right stretch tensor, $\mathbf{I}$ is the unit tensor and $\mathbf{R} = \mathbf{I}$ is the (orthogonal) rotation tensor and a dot denotes scalar product between two tensors.

The displacement gradient in the reference configuration is given by

$$\mathbf{H} = \mathbf{F} - \mathbf{I}; \qquad \mathbf{H} = \begin{bmatrix} -\nu\epsilon & 0 & 0 \\ 0 & -\nu\epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} . \tag{2.4}$$

The Jaumann or engineering strain tensor $\boldsymbol{\varepsilon}$ is defined in the reference configuration as

$$\boldsymbol{\varepsilon} = \mathbf{U} - \mathbf{I}; \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} -\nu\epsilon & 0 & 0 \\ 0 & -\nu\epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} . \tag{2.5}$$

Due to the rotation-free deformation of the beam, the Jaumann strain tensor is equivalent to the displacement gradient in the reference configuration, i.e. $\mathbf{H} \equiv \boldsymbol{\varepsilon}$.

The Green-Lagrange strain tensor $\mathbf{E}^0$ is defined in the reference configuration as

$$\mathbf{E}^0 = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) . \tag{2.6}$$

In view of (2.1), $\mathbf{E}^0$ takes the form

$$\mathbf{E}^0 = \begin{bmatrix} -\nu\epsilon + \frac{1}{2}\nu^2\epsilon^2 & 0 & 0 \\ 0 & -\nu\epsilon + \frac{1}{2}\nu^2\epsilon^2 & 0 \\ 0 & 0 & \epsilon + \frac{1}{2}\epsilon^2 \end{bmatrix} . \tag{2.7}$$

The Almansi-Euler strain tensor $\mathbf{E}$ is defined in the current configuration as

$$\mathbf{E} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) . \tag{2.8}$$

Taking into account that the Almansi-Euler and Green-Lagrange strain tensors are related to each other through

$$\mathbf{E} = \mathbf{F}^{-T} \cdot \mathbf{E}^0 \cdot \mathbf{F}^{-1} \tag{2.9}$$

and recalling that the reference and current configurations of the beam are investigated now in the same Cartesian frame, the Almansi-Euler strain components can be expressed in terms of the Green-Lagrange strain components as

$$E_{ij} = \begin{cases} \dfrac{1}{(F_{ij})^2} E^0_{ij} & \text{if } \ i = j \ , \\ \\ 0 & \text{if } \ i \neq j \ , \end{cases} \tag{2.10}$$

i.e. the non-zero components of $\mathbf{E}$ are given by

$$E_{11} = \frac{1}{(1-\nu\epsilon)^2}\, E^0_{11}, \qquad E_{22} = \frac{1}{(1-\nu\epsilon)^2}\, E^0_{22}, \qquad E_{33} = \frac{1}{(1+\epsilon)^2}\, E^0_{33} . \tag{2.11}$$

As is well known, for stretches close to zero the Green-Lagrange and Almansi-Euler strain tensors become identical with the engineering or Jaumann strain tensor. Note that independently of how large ϵ is, the structure of the Jaumann strain tensor is the same as that of the infinitesimal strain tensor.

3. Stress measures

Four important stress measures are considered here for the beam under uniaxial tension: the Cauchy (or true) stress tensor, $\mathbf{S}$, defined in the current configuration, the first Piola-Kirchhoff stress tensor, $\mathbf{T}$, which is a two-point tensor, the second Piola-Kirchhoff stress tensor, $\mathbf{S}^0$, defined in the reference configuration, and the less known, though very important, Biot stress tensor, $\boldsymbol{\sigma}$, defined in the reference configuration [2].

The relationships between these stress measures are well known and can be found in many books on continuum mechanics (see e.g.[3]): assuming that the Cauchy stress tensor $\mathbf{S}$ is known, the first Piola-Kirchhoff stress tensor is obtained as

$$\mathbf{T} = J\,\mathbf{S}\cdot\mathbf{F}^{-T}\ , \tag{3.1}$$

where J is the Jacobian of the deformation gradient, and the second Piola-Kirchhoff stress tensor as

$$\mathbf{S}^0 = \mathbf{F}^{-1}\cdot\mathbf{T} = J\,\mathbf{F}^{-1}\cdot\mathbf{S}\cdot\mathbf{F}^{-T}\ . \tag{3.2}$$

The Biot stress tensor $\boldsymbol{\sigma}$ is obtained through the polar decomposition of the first Piola-Kirchhoff stress tensor

$$\mathbf{T} = \mathbf{R}\cdot\boldsymbol{\sigma}\ , \qquad\qquad \boldsymbol{\sigma} = \mathbf{R}^T\cdot\mathbf{T}\ , \tag{3.3}$$

where $\mathbf{R}$ is the proper orthogonal rotation tensor obtained from the polar decomposition of the deformation gradient. The Biot stress tensor is generally nonsymmetric. For isotropic materials $\boldsymbol{\sigma}$ becomes coaxial with $\mathbf{U}$ and is, therefore, symmetric.

For the isotropic beam under uniaxial tension, each stress tensor introduced above has only one non-zero component which is the normal stress in the axial (z or Z) direction. In addition, since the deformation is rotation-free, the first Piola-Kirchhoff stress tensor is identical with the Biot stress tensor, i.e.

$$\mathbf{T} = \mathbf{R}\cdot\boldsymbol{\sigma} = \mathbf{I}\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}\ . \tag{3.4}$$

The only non-zero component of the first Piola-Kirchhoff as well as the Biot stress tensor is the nominal stress σ, which is the quotient of the axial force and the cross sectional area of the undeformed beam:

$$T_{33} = \sigma_{33} = \sigma = \frac{F}{A_0}\ . \tag{3.5}$$

The non-zero components of the two other stress tensors can be obtained by applying the transformation formulas (3.1) and (3.2). Taking into account that the Jacobian of the deformation (2.1) is

$$J = \det\,\mathbf{F} = (1+\epsilon)(1-\nu\epsilon)^2\ , \tag{3.6}$$

the Cauchy normal stress in the z direction of the beam is

$$S_{33} = \frac{1}{(1-\nu\epsilon)^2}\, T_{33} = \frac{1}{(1-\nu\epsilon)^2}\, \sigma \tag{3.7}$$

and the second Piola-Kirchhoff normal stress in the Z direction of the beam is

$$S^0_{33} = \frac{1}{1+\epsilon}\, T_{33} = \frac{1}{1+\epsilon}\, \sigma\ . \tag{3.8}$$

The first Piola-Kirchhoff-, the Biot- , the Cauchy- and the second Piola-Kirchhoff stress tensors for the beam are given by

$$\mathbf{T} \equiv \boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma \end{bmatrix}\ , \tag{3.9}$$

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma\,(1-\nu\epsilon)^{-2} \end{bmatrix};\quad \mathbf{S}^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma\,(1+\epsilon)^{-1} \end{bmatrix}\ . \tag{3.10}$$

The relationship between the Cauchy normal stress S_{33} and the second Piola-Kirchhoff normal stress S^0_{33} can be obtained from (3.7)-(3.8):

$$S_{33} = \frac{1+\epsilon}{(1-\nu\epsilon)^2}\, S^0_{33}\ . \tag{3.11}$$

4. Relations between conjugate stress and strain measures

The strain energy density of a hyperelastic body can be expressed by the inner product of different, appropriate stress and strain tensors. We are interested here in the relations between the following work-conjugate stress and strain measures:

– second Piola-Kirchhoff stress tensor ⇔ Green-Lagrange strain tensor
– first Piola-Kirchhoff stress tensor ⇔ displacement gradient tensor
– Cauchy stress tensor ⇔ Almansi-Euler strain tensor
– Biot stress tensor ⇔ Jaumann strain tensor.

As mentioned in the Introduction, the relation between the nominal stress and the stretch of the beam under uniaxial tension is considered to be linear and this relation is expressed by Hooke's law:

$$\sigma = E_Y\, \epsilon\ , \tag{4.1}$$

where E_Y is the Young's or elasticity modulus of the material of the beam and ϵ is not necessarily infinitesimal. In view of the previous sections, the Hooke's law between the different work-conjugate stress and strain components of the beam can easily be derived. Taking into account that – according to (3.5) – the nominal stress σ is equal to the first Piola-Kirchhoff normal stress T_{33} as well as the Biot normal stress σ_{33} in the axial direction, and also that the stretch of the beam ϵ is equal to the displacement gradient component H_{33} as well as the Jaumann normal strain ε_{33} in the axial direction, we obtain the following stress-strain relationships between the appropriate conjugate stress and strain components:

I. Piola-Kirchhoff stress – displacement gradient:

$$T_{33} = E_Y \, H_{33} \tag{4.2}$$

Biot stress – Jaumann strain:

$$\sigma_{33} = E_Y \, \varepsilon_{33} \tag{4.3}$$

Cauchy stress – Almansi-Euler strain:

$$S_{33} = \frac{2(1+\epsilon)^2}{(2+\epsilon)(1-\nu\epsilon)^2} E_Y \, E_{33} = E_Y^{\mathrm{cur}} \, E_{33} \tag{4.4}$$

II. Piola-Kirchhoff stress – Green-Lagrange strain:

$$S^0_{33} = \frac{2}{(1+\epsilon)(2+\epsilon)} E_Y \, E^0_{33} = E_Y^{\mathrm{ref}} \, E^0_{33} \tag{4.5}$$

where

$$E_Y^{\mathrm{cur}}(\epsilon) = \frac{2(1+\epsilon)^2}{(2+\epsilon)(1-\nu\epsilon)^2} E_Y \tag{4.6}$$

and

$$E_Y^{\mathrm{ref}}(\epsilon) = \frac{2}{(1+\epsilon)(2+\epsilon)} E_Y \tag{4.7}$$

are the actual moduli of elasticity in the current and reference configurations, respectively. It can be seen from (4.2) and (4.3) that, independently of how large the stretch of the beam is, the I. Piola-Kirchhoff normal stress T_{33} and the displacement gradient component H_{33} as well as the Biot normal stress σ_{33} and the Jaumann strain ε_{33} are related to each other by the same (known and constant) Young's modulus E_Y. This is not the case, however, for either the Cauchy normal stress S_{33} and Almansi-Euler strain E_{33} or the II. Piola-Kirchhoff stress S^0_{33} and Green-Lagrange strain E^0_{33}. They are related to each other through the 'modified' Young's moduli E_Y^{cur} and E_Y^{ref}, respectively, and, as it is seen from (4.6) and (4.7), neither E_Y^{cur} nor E_Y^{ref} is constant with respect to the stretch ϵ. The functions $E_Y^{\mathrm{cur}}(\epsilon)$ and $E_Y^{\mathrm{ref}}(\epsilon)$ for stretch values $0 \leq \epsilon \leq 0.5$ are shown in Figure 1, assuming that the measured Young's modulus E_Y is constant and unity.

Relative errors in the Cauchy stress (at $\nu = 0.0$) as well as in the II. Piola-Kirchhoff stress for relatively small stretch values are given in Table 1 with the assumption that the measured constant Young's modulus E_Y is used instead of the actual moduli $E_Y^{\mathrm{cur}}$and E_Y^{ref}. As expected, for very small stretches the error is not significant; approximately 1% error is obtained in both stresses when the stretch ϵ attains the value of 0.0065.

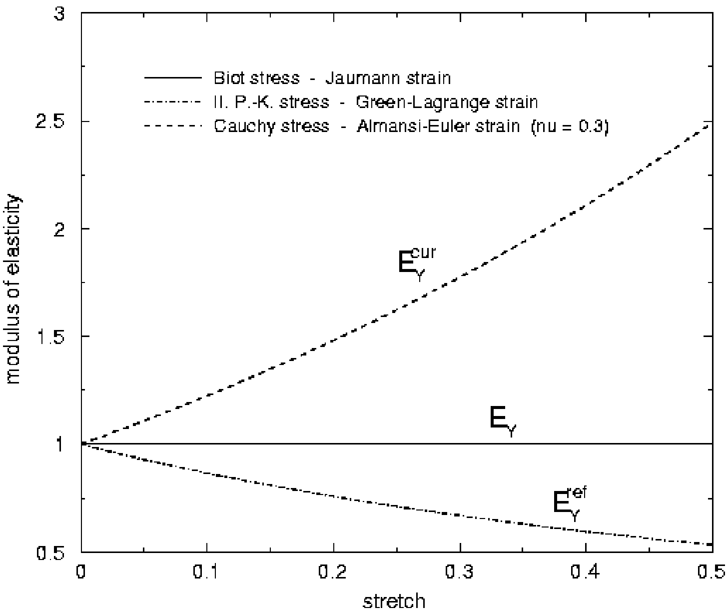

Figure 1. Modulus of elasticity for different conjugate stress and strain measures

ϵ	rel. error in S_{33} (%)	rel. error in S^0_{33} (%)
0.0001	0.015	0.015
0.0005	0.075	0.075
0.001	0.15	0.15
0.005	0.75	0.75
0.01	1.50	1.50
0.05	7.56	7.56
0.1	15.24	13.42

Table 1. Relative error in the Cauchy stress (at $\nu = 0.0$) and in the II. Piola-Kirchhoff stress

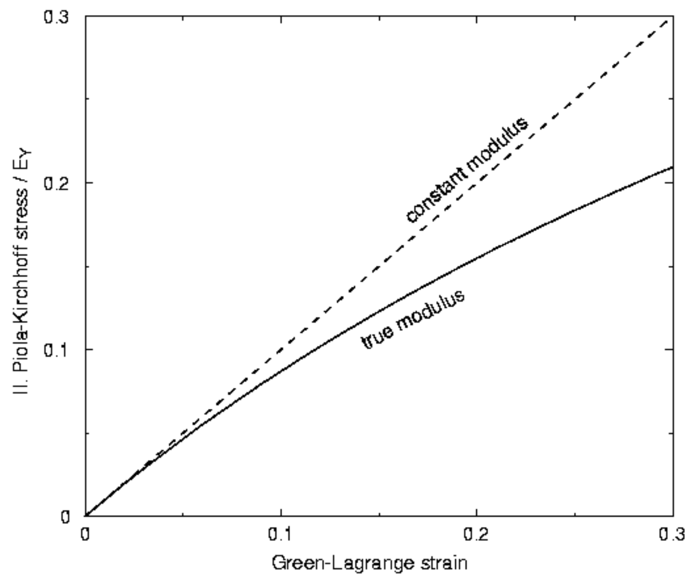

Figure 2. II. Piola-Kirchhoff stress against the Green-Lagrange strain with E_Y fixed

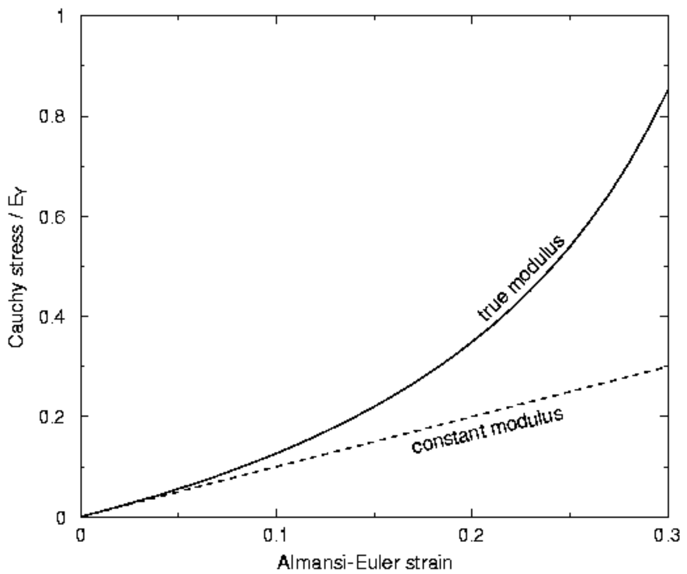

Figure 3. Cauchy stress against the Almansi-Euler strain with E_Y fixed

Taking into account that the stretch ϵ can be expressed by the Green-Lagrange strain E^0_{33} as [3]

$$\epsilon = \sqrt{2E^0_{33} + 1} - 1 \tag{4.8}$$

and inserting (4.8) in (4.5), the II. Piola-Kirchhoff stress S^0_{33} can be obtained directly in terms of the Green-Lagrange strain E^0_{33}. The $S^0_{33}(E^0_{33})$ function is plotted in Figure 2 for Green-Lagrange strain values $0 \leq E^0_{33} \leq 0.3$. The Cauchy stress S_{33} can also be expressed directly in terms of the Almansi-Euler strain E_{33} by utilizing the fact that stretch ϵ can be expressed by the Almansi-Euler strain E_{33} as [3]

$$\epsilon = \frac{1}{\sqrt{1 - 2E_{33}}} - 1\,. \tag{4.9}$$

Inserting (4.9) in (4.4), the function $S_{33}(E_{33})$ is obtained which is shown in Figure 3 for Almansi-Euler strain values $0 \leq E_{33} \leq 0.3$. Dashed lines in both Figures 2 and 3 indicate the linear (incorrect) stress-strain curves when the constant Young's modulus E_Y is used for obtaining the II. Piola-Kirchhoff stresses from the Green-Lagrange strains and the Cauchy stresses from the Euler-Almansi strains.

5. Conclusions

In advanced materials it is not unusual that the limit of the linearly elastic behavior in terms of stretches is much higher than for classical materials like metals. Independently of the stress and strain measures applied by the underlying formulation, an accurate numerical analysis of geometrically nonlinear problems involving materials of that kind requires the use of correct elasticity modulus in the stress-strain relations at different stretch and strain levels.

Considering a homogeneous isotropic prismatic beam under uniaxial tension, the Young's modulus of the linearly elastic material is the tangent of the nominal stress versus stretch function. Independently of how large the stretches are, the constant Young's modulus measured that way can always be applied between the Biot stresses and Jaumann strains, being work-conjugate engineering stress and strain measures. (The I. Piola-Kirchhoff stress and displacement gradient components can also be related to each other through that constant Young's modulus, provided the deformation is rotation-free.) The constant Young's modulus cannot, however, be used for relating other stress and strain measures, such as the widely used second Piola-Kirchhoff stresses and Green-Lagrange strains or the Cauchy stresses and Euler-Almansi strains, without the restriction that the stretches in the material should be very small. This paper investigated the error resulting from the use of a constant Young's modulus for relating the above mentioned different conjugate stress and strain measures at different stretch levels.

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