A SHORT REVIEW ON THE BUCKLING OF COMPRESSED COLUMNS

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Abstract
The present review paper intends to briefly summarize a broad range of research on various stability problems of compressed beams. It is noted that multiple analysis approaches have been used, many are closed-form analytical, while others are classified as semi-analytical or simply numerical methods. This study overviews some of the most important theories and solutions for beam buckling from the 18th century up to date, bringing together the findings from several scientific perspectives. The emphasis is put on the recent novel results.

Keywords: *review article, buckling, beam, elasticity*

1. Introduction

One of the most essential features of engineering structures is buckling, which is the point where failure can occur. Buckling must be avoided when designing engineering structures. In many engineering systems, beams are common structural elements. A bunch of beam problems have been investigated recently – see, e.g, (Ecsedi and Gönczi, 2015; Ecsedi et al., 2021; Gurumoorthy and Rao, 2019; Ecsedi and Baksa, 2021). When a straight beam is loaded with an axially compressive force, it exhibits minor deformations until it reaches a critical condition known as buckling at the critical load value. In the literature, the term critical buckling load is typically used to refer to the lowest buckling load. In the domains of mechanical, structural, and aeronautical engineering, buckling of beams subjected to compressive load is a frequent issue. For compressed members that are the subject of numerous investigations, including static, dynamic, and stability tests, the computation of critical buckling load plays a significant role. Buckling first takes place about the axis having a minimum radius of gyration or least moment of inertia. So far, a lot of relevant research results have been achieved. This article intends to give a brief review to gather a bunch of major findings together.

Since beam buckling can be a prevalent cause of failure in engineering applications, it has been in the focus of research for a long time. The Swiss mathematician Leonhard Euler was a pioneer in this subject, publishing his well-known formula for the critical (buckling) load of straight bars under compression in 1759 (Euler, 1759). He also showed that there is another criterion for column strength that is unrelated to the material's crushing or yielding. Beam buckling is the loss of stability that is generally unrelated to material strength and this loss of stability usually happens within the material's elastic range. Since the stress in the column remains in elastic state, Euler's theory of column buckling is utilized to calculate the critical buckling load. When a column is about to buckle, the critical buckling load is the greatest load it can sustain. A simple formula for determining the critical buckling load for a homogeneous and perfectly straight column with no lateral pressure was developed by Euler and the following equation depicts Euler's famous buckling formula: $P_{crit} = \pi IE/(kL)^2$. Here P_{crit} is the critical (or buckling) load, L denotes column length, k denotes the effective length factor, and EI denotes the cross-sectional flexural stiffness. The fact that the material's compressive strength has no influence in calculating the critical buckling load is an intriguing observation that can be drawn from this calculation. Since then, the Euler buckling calculation has been important in physics and engineering for the stability and mechanical properties of slender structures ranging from nano to macrostructures (Timoshenko and Gere, 1961; Niklas, 1992). Certain assumptions about the point of axial load application, column material, cross-section, stress limits, and column failure underpin Euler's theory.

2. Major scientific findings of the previous century

Buckling research has become increasingly significant in engineering structures during the previous century. There is a long list of articles on buckling of beams. In paper (Shu, 1998) for beams with twin delaminations, a precise buckling analysis was done. After some threshold values of either the depth or the length of the delamination, the buckling load's sensitivity to either the depth or the length of the delamina- tion grew fast. The author checked the results of the critical loads obtained in his research with those published by Simitses et al (Simitses et al., 1985), and Lim and Parsons (Lim and Parsons, 1993) using either finite element method (FEM) or a new energy approach to validate the correctness of the model used in his work. Using two distinct algorithms based on the Rayleigh-Ritz method in the paper (Lee and Ng, 1994), the critical buckling loads of simply supported beams with a step in the middle have been reported. The second algorithm produced smaller critical buckling loads for beams with large variations in thickness across a step ($d_2/d_1 = 0.1...0.5$ or $2.0...10.0$).

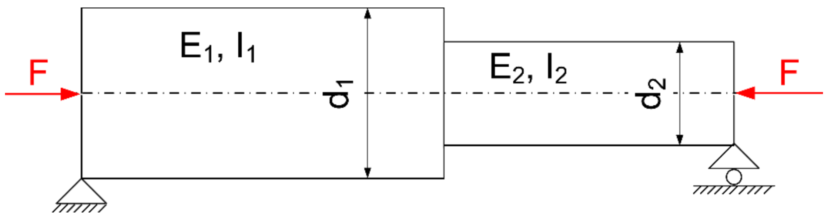

Figure 1. *A simply supported beam with a step* (Lee and Ng, 1994)

Khdeir & Reddy (Khdeir and Redd, 1997) studied the buckling behavior of cross-ply rectangular beams with arbitrary boundary conditions using analytical solutions of revised classical beam theories. The relationship between critical buckling loads and geometrical parameters as well as support conditions has been studied and they supposed that, as in the material principal axes, all layers were considered to have the same thickness and orthotropic material properties. Their numerical results show that the classical beam theory (CBT) overpredicts buckling loads when compared to shear deformation theories, and this discrepancy diminished as the length-to-thickness ratio increased. For multiple boundary conditions, Eisenberg (Eisenberger, 1991) developed an accurate solution for the buckling loads of variable cross-section columns loaded by variable axial force. The buckling load, according to his study, is defined as the force that makes the determinant of the stiffness matrix equal to zero. To show the accuracy and flexibility of his technique, several examples from the literature were offered and compared to his published results.

Jensen (Jensen, 2000) performed a non-linear buckling study of an elastic beam under an axial static force and high-frequency axial excitation. To generate a set of autonomous model equations regulating the slow averaged behavior, a Galerkin beam discretization was utilized, and the approach of direct partition of motion was adopted. With more high-frequency excitation, the buckling stress rised, according to his research. Authors of (Abramovich et al., 1996) studied the buckling behavior of nonsymmetric laminated composite beams using a first-order shear deformation theory. The exact element method was successfully applied to calculate buckling loads. A parametric study was performed to investigate the influence of boundary conditions, materials, number of layers, on the buckling loads of rectangular, cross-ply laminated composite beams. Eisenberger et al. (Eisenberger et al., 1986) used elastic and geometric stiffness matrices for beams on an elastic foundation to find the buckling loads and mode shapes. The critical loads calculated using their methodology are slightly better than the precise ones obtained using the geometric stiffness approach. They advised at that time to include their approach into typical beam stability computer systems, which would increase their capabilities while also lowering data preparation time.

3. On research progress from recent decades

Due to the fact that non-uniform and inhomogeneous beams may give an efficient option to bear the applied greater compressive stresses in engineering structures, there has been a lot of written works on investigating their stability up till now. With ten various combinations of end conditions, the transfer matrix method was used to investigate the buckling of multi-step nonuniform beams (Figure 2) with elastically restrained boundary conditions subjected to several concentrated axial forces in (Li, 2001). The author proved that a beam on an elastic foundation can be treated as a beam with translational spring supports at intermediate points. The critical load findings obtained using a transfer matrix method were compared to those obtained using the finite element method and they were in good agreement. Aydogdu (Aydogdu, 2006) investigated the buckling of cross-ply laminated beams under various boundary circumstances. The critical buckling loads were calculated using the Ritz method, which involved expressing the three kinematic unknown components as a set of simple algebraic polynomials. He concluded that, for the identical length-to-thickness ratios, the non-dimensional critical buckling load of beams subjected to clamped supported boundary conditions took the maximum values, whereas those subjected to clamped-free took the minimum values.

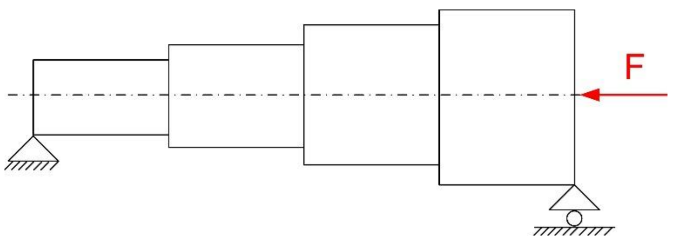

Figure 2. *A multi-stepped beam in compression* (Li, 2001)

The crtical forces and mode-shape functions of Euler-Bernoulli beams with an arbitrary number of non-breathing fractures subjected to a constant axial force were reported in (Aydin, 2008) by Aydin.

According to his results, the presence of a crack significantly reduced the axial load level needed to initiate buckling. A simple beam with a crack depth *50%* of its height, for example, would buckle on only *63%* of the Euler load. The critical load level was further reduced when the crack depth or the number of cracks increased. Stojanovic et al. (Stojanovic et al., 2011) investigated the buckling of an elastically coupled simply supported double beam under compressive axial loading using the Euler-Bernoulli beam theory (see Figure 3). They demonstrated that the critical buckling load reduced as the axial load ratio of the lower beam and the upper beam grew, but it increased when the stiffness modulus *K* of the Winkler elastic foundation layer increased. The buckling instability of a system of three simply supported elastic Timoshenko beams, connected together by Winkler elastic layers, and each beam subjected to the identical compressive axial force, was investigated in paper (Stojanovic et al., 2012) – see Figure 4. Authors observed that when the stiffness modulus increased, the critical buckling load increased, but it reduced as the vibration mode increased.

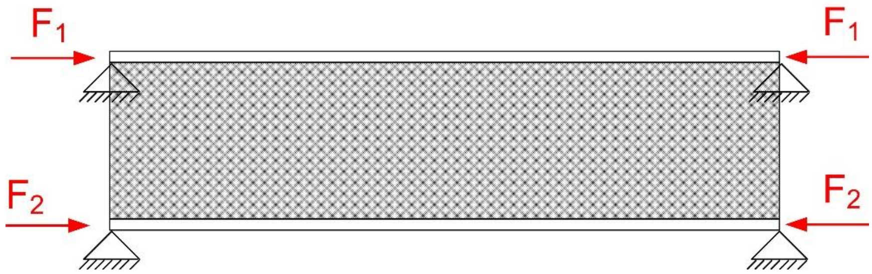

Figure 3. *Double-beam complex system* (Stojanovic et al., 2011)

The buckling of nanocomposite beams reinforced by single-walled carbon nanotubes resting on an elastic foundation were investigated in (Yas and Samadi, 2012) using the Timoshenko beam theory. The Hamilton's principle was used by the author to determine the equations of motion. The critical buckling load of carbon nanotube reinforced composite beams were calculated using the generalized differential quadrature method for various boundary conditions. Clamped-clamped beams had the highest critical buckling loads, followed by clamped-hinged, hinged-hinged, and finally, clamped-free supports, according to the results of the inquiry into the effect of boundary conditions. Rychlewska (Rychlewska, 2014) demonstrated a method for calculating the critical buckling load of axially functionally graded beams subjected to a distributed axial load for clamped-clamped, pinned-pinned and clamped-pinned beams. She hypothesized that changes in the flexural stiffness ($I(x)E(x) = D\ exp(2\beta\pi/L)$) as well as in the axial load ($P(x) = P\ exp(2\beta\pi/L)$ on the beam follow an exponential distribution. It was observed that for clamped-pinned beams an increase in the real number β causes a decrease in the critical buckling load. Elishakoff (Elishakoff, 2012) investigated the buckling of axially functionally graded material columns. The goal of his research was to identify a polynomial variation of the modulus of elasticity *E* such that the buckling value exceeded in the case of a cantilever column with a constant cross-sectional area.

The buckling analysis of an edge cracked functionally graded cantilever Timoshenko beam under axial compressive loads was performed in the study (Akbas, 2015). It was assumed that the cracked beam was modeled as a connection between two sub-beams via a massless elastic rotational spring. It has been discovered that employing FGM can reduce the harmful impacts of buckling loads in the presence of cracks, i.e., the undesirable impacts of the crack can be reduced by selecting the appropriate ratio of material properties.

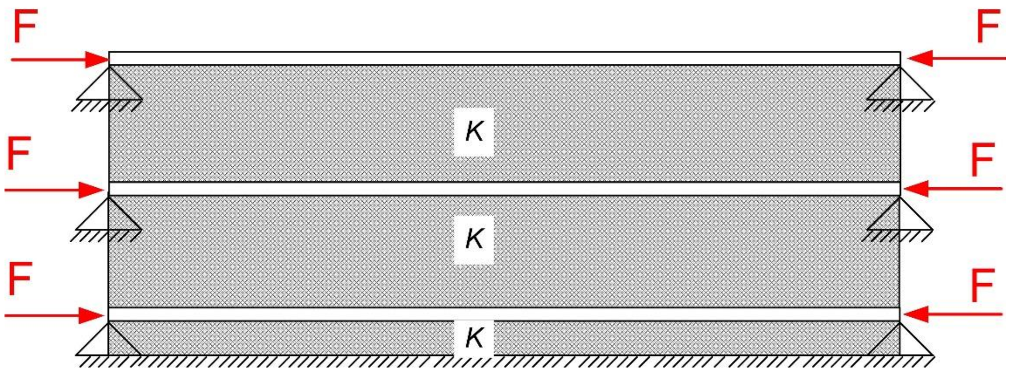

Figure 4. *Three Timoshenko beams connected with Winkler layers* (Stojanovic et al., 2012)

Anghel (Anghel, 2018) demonstrated critical buckling load calculations for a variety of beam scenarios with various basic boundary conditions – see Figure 5. Green's functions were used to convert the differential equations governing the deflection behavior of such beams into integral form. He calculated numerically the Green's functions using a particular integral approach of strength of materials, taking into account that displacements in certain locations of a beam were caused by unit forces applied in other points. When he compared the numerical critical buckling loads with the analytical data acquired from the literature review, his work demonstrated good agreement from an engineering standpoint.

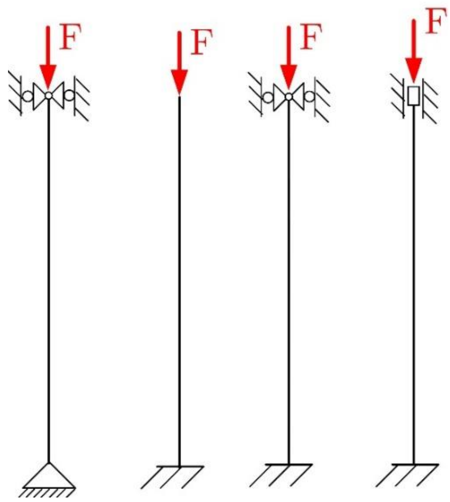

Figure 5. *Some investigated support arrangements in* (Anghel, 2018)

Messaoudi et al. (Kiss et al., 2022) presented a novel solution procedure to the stability problem of various heterogeneous beams with three supports. The end restraints are supplemented with an intermediate roller support in their investigations. Some setups are shown in Figure 6. They used a Green function technique of the three-point boundary value problem that describes the mechanical behavior of such beams to transform the linear stability problem into an eigenvalue problem governed by a homogeneous Fredholm integral equation. The eigenvalue problem was solved using the boundary element method where the critical load was the sought parameter. The maximum critical load has been found when the roller support located in the middle. With the use of the exact transfer matrix, Yldrm (Yildirim, 2021) investigated the buckling of rectangular beams with ceramic liners at the top and bottom surfaces. For some boundary circumstances, the dimensionless buckling loads of these beams were numerically estimated. The impact of the thickness of the ceramic liner on buckling loads was also researched by the author, and it was discovered that a ceramic liner significantly improves buckling loads.

Authors of article (Huang and Luo, 2011) presented a new and simple approach to solve buckling of axially inhomogeneous beams with a continuous elastic restraint. They converted the governing equation with varying coefficients to linear algebraic equations for varied end boundary conditions and then a characteristic equations in critical buckling loads were obtained. They showed how the gradient parameter (which describes the change in volume fraction of both constituents involved) affected the critical buckling forces of an Al/ZrO_2 composite beams under different end supports. They determined that when the gradient parameter increased, the critical buckling loads dropped, indicating that the gradient parameter is the most important factor in determining the buckling loads, despite the fact that two component phases remain unchanged. Critical buckling loads for stepped beams clamped at one end and elastically fixed at the other were determined in paper (Lellep and Kraav, 2016). The effect of a crack on the stability of a beam was described as a change in the local flexibility of the beam by combining the methods of elastic beam theory and linear elastic fracture mechanics. They discovered that fractures had a significant impact on the critical buckling load, particularly when large cracks were present. However, studies revealed that minor cracks that penetrate less than *10%* of the thickness had little impact on the structure's stability.

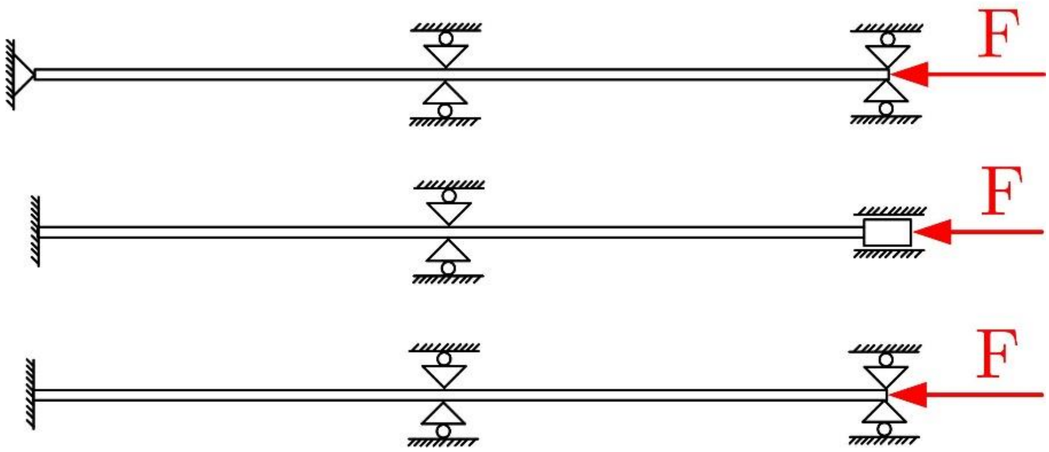

Figure 6. *Buckling of a straight beams with three supports* (Kiss et al., 2022)

4. Conclusions

Some models and solutions for buckling of beams that are available in the literature were overviewed in this work. There were many types of beams studied, each with its own set of boundary conditions. In the literature, there is still a lot of interest in the buckling beams and their applications, as evidenced by review publications. Some aspects of the formulation of the nonlinear buckling issue were mentioned thus opening the way for more effective application of buckled beam theories.

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