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SOLUTION OF THE SAINT-VENANT TORSION OF ORTHOTROPIC BAR BY THE METHOD OF BOUNDARY COLLOCATION

István Ecsedi
Professor Emeritus, Institute of Applied Mechanics, University of Miskolc
3515 Miskolc, Miskolc-Egyetemváros, e-mail: mechecs@uni-miskolc.hu

Ákos József Lengyel
Senior lecturer, Institute of Applied Mechanics, University of Miskolc
3515 Miskolc, Miskolc-Egyetemváros, e-mail: mechlen@uni-miskolc.hu

Attila Baksa
Associate professor, Institute of Applied Mechanics, University of Miskolc
3515 Miskolc, Miskolc-Egyetemváros, e-mail: mechab@uni-miskolc.hu

Dávid Gönczi
Senior lecturer, Institute of Applied Mechanics, University of Miskolc
3515 Miskolc, Miskolc-Egyetemváros, e-mail: mechgoda@uni-miskolc.hu

Abstract
This paper deals with an approximate solution of Saint-Venant torsion problem of orthotropic bar with solid cross section. The boundary collocation is used to get the approximate analytical solution of the torsion problem. Examples illustrate the applications of the presented numerical solution.

Keywords: *boundary collocation, Saint-Venant torsion, orthotropic bar, torsion function, Prandtl's stress function*

1. Introduction

The boundary collocation method is a method for the numerical solution of ordinary differential equations and partial differential equations. The concept is to choose a finite dimensional space of the candidate solutions which satisfy the differential equations but they do not satisfy the prescribed boundary conditions. The linear combination of the candidate solutions satisfies the boundary conditions only on the selected points of the boundary curve or boundary surface. These selected points are the collocation points.

This paper deals with two-dimensional problem which can be described by an elliptical partial differential equation. The linearly independent solutions of the differential equation are generated by source points which are outside of the considered simply connected domain in our case. In this study the number of source points and number of collocation boundary points are the same.

The general method of boundary collocation is formulated in book by Kołodziey and Zieliński (Kołodziej and Zieliński, 2009) and book by Li et al. (Li et al., 2008). Yuanhan (Yuanhan, 1990) used the boundary collocation method to obtain the torsional rigidity of a thick-walled cylinder with an extremal radial crack. Paper by Mierzwiczak and Kołodziej (Mierzwiczak and Kołodziej, 2012) presents a comparison of different methods of choosing collocation points for boundary collocation method. The goal is to find the optimal positions for the source points (Mierzwiczak and Kołodziej, 2012). A torsion of isotropic homogeneous triangular bar with circular hole cross section is investigated that how the position and number of source points affect the accuracy of the solution (Mierzwiczak and Kołodziej, 2012). Paper by Kołodziej and Fraska gives the solution of torsional problem of regular polygonal cross section by means of boundary collocation method. The result obtained is useful in the design of torsional members (Kołodziej and Fraska, 2005). The numerical experiments related with the shape of the source contour in application of the boundary collocation method to the elastic torsion of prismatic bars are considered in paper by Gorzelańczyk and Kołodziej (Gorzelańczyk and Kołodziej, 2008). The Saint-Venant torsion for five different cross sections is solved: L-section, [-section, +-section, ⅃-section and I-section by means of the method of boundary collocation in (Gorzelańczyk and Kołodziej, 2008).

In this paper the Saint-Venant torsion problem of Cartesian orthotropic homogeneous bar with solid cross section is considered. The cross section of the bar is shown in Fig. 1. A denotes the cross section, ∂A is the boundary curve of the cross section. The shear moduli of the elastic orthotropic material of the bar are $G_1 = G_{xz}$ and $G_2 = G_{yz}$ and the applied torque is T and the rate of twist is ϑ. The torsional rigidity S of the cross section is defined as (Lekhnitskii, 1981; Lekhnitskii, 1971)

$$S = \frac{T}{\vartheta}. \tag{1}$$

2. Governing equations

2.1. Formulation of torsion problem by torsion function

The torsion function of orthotropic elastic bar is denoted by $\omega = \omega(x, y)$. It is known that the torsion function is the solution of the following boundary value problem (Lekhnitskii, 1981; Lekhnitskii, 1971; Sokolnikoff, 1956)

$$G_1 \frac{\partial^2 \omega}{\partial x^2} + G_2 \frac{\partial^2 \omega}{\partial y^2} = 0, \quad (x, y) \in A, \tag{2}$$

$$n_x G_1 \left(\frac{\partial \omega}{\partial x} - y \right) + n_y G_2 \left(\frac{\partial \omega}{\partial y} + x \right) = 0, \quad (x, y) \in \partial A. \tag{3}$$

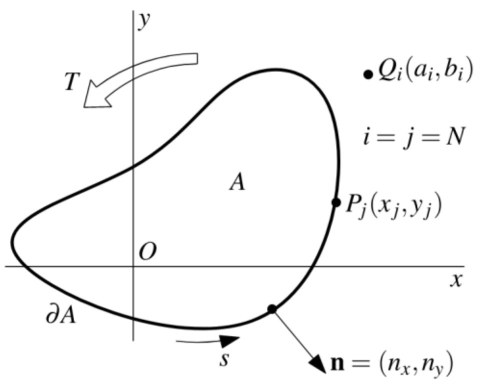

Figure 1. *Cross section of the orthotropic bar*

In equation (3) n_x, n_y are the components of unit normal vector of the boundary curve ∂A (Fig. 1).

The shearing stresses τ_{xz} and τ_{yz} can be computed as (Lekhnitskii, 1981; Lekhnitskii, 1971; Sokolnikoff, 1956)

$$\tau_{xz} = \vartheta G_1\left(\frac{\partial\omega}{\partial x} - y\right), \quad \tau_{yz} = \vartheta G_2\left(\frac{\partial\omega}{\partial y} + x\right). \tag{4}$$

The expression of the torsional rigidity S in term of $\omega = \omega(x, y)$ is as follows

$$S = \int_A \left[G_2\left(\frac{\partial\omega}{\partial y} + x\right)x - G_1\left(\frac{\partial\omega}{\partial x} - y\right)y \right]\mathrm{d}A. \tag{5}$$

From equation (5) it follows that

$$S = G_2 I_y + G_1 I_x + \int_A \left[G_2 \frac{\partial\omega}{\partial y} - G_1 \frac{\partial\omega}{\partial x} \right]\mathrm{d}A, \tag{6}$$

where

$$I_x = \int_A y^2 \mathrm{d}A, \quad I_y = \int_A x^2 \mathrm{d}A. \tag{7}$$

Application of Gauss's theorem in equation (6) gives

$$\int_A \left[G_2 \frac{\partial(x\omega)}{\partial y} - G_1 \frac{\partial(y\omega)}{\partial x} \right]\mathrm{d}A = \int_{\partial A} \omega\left[n_y G_2 x - n_x G_1 y \right]\mathrm{d}s. \tag{8}$$

In equation (8) s is an arclength coordinate defined on the boundary curve ∂A (Fig. 1). The combination of equation (6) with equations (7) and (8) yields to the result

$$S = G_1 I_x + G_2 I_y + \int_{\partial A} \omega\left[n_y G_2 x - n_x G_1 y \right]\mathrm{d}s. \tag{9}$$

It is very easy to check that the function

$$F_i(x, y) = \ln\left[\frac{G_0}{G_1 c^2}\left(x - a_i^2\right) + \frac{G_0}{G_2 c^2}\left(y - b_i^2\right) \right], \tag{10}$$

where $G_0 = \sqrt{G_1 G_2}$ and c is a constant with unit [length] satisfies the partial differential equation (2) for arbitrary value of c. The approximate solution of the boundary-value problem formulated in equations (2) and (3) is looked for as a linear combination of the fundamental solutions

$$\omega(x, y) = \sum_{i=1}^{N} q_i F_i(x, y). \tag{11}$$

The unknown constants q_i $(i = 1, 2..., N)$ are computed from the boundary condition (3)

$$n_{jx}G_1\left(\frac{\partial\omega}{\partial x}\right)_{\substack{x=x_j\\y=y_j}}+n_{jy}G_2\left(\frac{\partial\omega}{\partial y}\right)_{\substack{x=x_j\\y=y_j}}=n_{jx}y_jG_1-n_{jy}x_jG_2=0. \tag{12}$$

Simple computation shows that

$$n_xG_1\frac{\partial F_i}{\partial x}=\frac{2n_xG_1G_2(x-a_i)}{G_2(x-a_i)^2+G_1(y-b_i)^2}, \tag{13}$$

$$n_yG_2\frac{\partial F_i}{\partial x}=\frac{2n_yG_1G_2(y-b_i)}{G_2(x-a_i)^2+G_1(y-b_i)^2}. \tag{14}$$

Let A_{ij} be defined as

$$A_{ij}=\frac{2n_{jx}G_1G_2(x_j-a_i)}{G_2(x_j-a_i)^2+G_1(y_j-b_i)^2}+\frac{2n_{jy}G_1G_2(y_j-b_i)}{G_2(x_j-a_i)^2+G_1(y_j-b_i)^2},\quad (i,j=1,2,...,N). \tag{15}$$

We introduce the vector f_j

$$f_j=n_{jx}y_jG_1-n_{jy}x_jG_2,\quad (j=1,2,...,N). \tag{16}$$

From equation (12) we obtain a system of linear equations for the unknown coefficients q_i $(i=1,2,...,N)$

$$\sum_{i=1}^{n}A_{ij}q_i-b_j=0,\quad (j=1,2,...,N). \tag{17}$$

2.2. Formulation of torsional problem by Prandtl's stress function

It is known (Lekhnitskii, 1981; Lekhnitskii, 1971; Sokolnikoff, 1956) the Prandtl's stress function formulation of the torsion problem of orthotropic bar leads to the following Dirichlet type boundary-value problem

$$\frac{1}{G_2}\frac{\partial^2 U}{\partial x^2}+\frac{1}{G_1}\frac{\partial^2 U}{\partial y^2}=-2,\quad (x,y)\in A, \tag{18}$$

$$U(x,y)=0,\quad (x,y)\in\partial A. \tag{19}$$

The shearing stresses τ_{xz} and τ_{yz} in terms of $U=U(x,y)$ can be expressed as

$$\tau_{xz}=\vartheta\frac{\partial U}{\partial y},\quad \tau_{yz}=-\vartheta\frac{\partial U}{\partial x}. \tag{20}$$

The torsional rigidity of the orthotropic bar is (Lekhnitskii, 1981; Lekhnitskii, 1971; Sokolnikoff, 1956)

$$S = 2\int_A U \mathrm{d}A = \int_A \left(-y\frac{\partial U}{\partial y} - x\frac{\partial U}{\partial x} \right) \mathrm{d}A. \tag{21}$$

The Prandtl's stress function $U = U(x, y)$ can be represented as

$$U(x,y) = V(x,y) - \frac{G_2 x^2 + G_1 y^2}{2}. \tag{22}$$

It is evident that $V = V(x, y)$ is the solution of the following boundary value problem

$$G_1 \frac{\partial^2 V}{\partial x^2} + G_2 \frac{\partial^2 V}{\partial y^2} = 0, \quad (x, y) \in A, \tag{23}$$

$$V(x,y) = \frac{G_2 x^2 + G_1 y^2}{2}, \quad (x, y) \in \partial A. \tag{24}$$

The approximate solution of the boundary-value problem defined by equations (23), (24) is searched by the application of boundary collocation method as

$$V(x,y) = \sum_{i=1}^{n} p_i F_i(x,y), \tag{25}$$

where the unknown constants p_i $(i = 1, 2, ..., N)$ are obtained from the boundary condition (24) that is

$$\sum_{i=1}^{n} p_i F_i(x_j, y_j) = \frac{G_2 x_j^2 + G_1 y_j^2}{2}. \tag{26}$$

A detailed form of equation (26) is

$$\sum_{i=1}^{N} A_{ij} p_i - b_j = 0, \quad (j = 1, 2, ..., N). \tag{27}$$

Here,

$$A_{ij} = \ln\left[\frac{G_0}{G_1 c^2}(x_j - a_i)^2 + \frac{G_0}{G_2 c^2}(y_j - b_i)^2 \right], \tag{28}$$

$$b_j = \frac{G_2 x_j^2 + G_1 y_j^2}{2}, \quad (i, j = 1, 2, ..., N). \tag{29}$$

Approximate formulae of shearing stresses are as follows

$$\tau_{xz} = \vartheta \left\{ \sum_{i=1}^{N} p_i \frac{2G_1 (y - b_i)}{G_2 (x - a_i)^2 + G_1 (y - b_i)^2} - G_1 y \right\}, \tag{30}$$

$$\tau_{yz} = \vartheta \left\{ -\sum_{i=1}^{N} p_i \frac{2G_2 (x - a_i)}{G_2 (x - a_i)^2 + G_1 (y - b_i)^2} + G_2 x \right\}. \tag{31}$$

Approximate formula for the torsional rigidity is obtained from equation (21)

$$S = G_1 J_x + G_2 J_y - 2\sum_{i=1}^{N} p_i \int_A \frac{G_2(x^2 - xa_i) + G_1(y^2 - yb_i)}{G_2(x-a_i)^2 + G_1(y-b_i)^2} \mathrm{d}A. \tag{32}$$

3. Example

3.1. Solid circular cross section

The radius of the boundary circle of cross section is R and the distance of source points from the boundary circle is r (Fig. 2). Exact solution of the orthotropic solid circular cross section is known

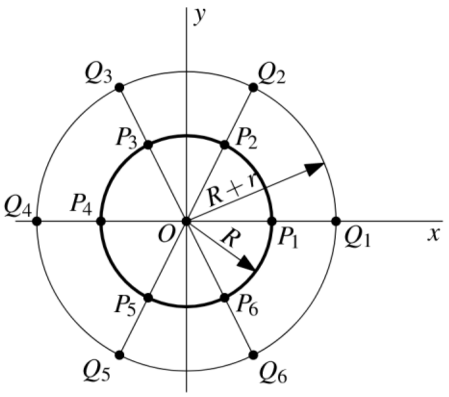

Figure 2. *Solid circular cross section whose boundary curve a circle with radius R*

$$\omega(x,y) = \frac{G_1 - G_2}{G_1 + G_2} xy, \tag{33}$$

$$\tau_{xz} = -\vartheta \frac{2G_1 G_2}{G_1 + G_2} y, \tag{34}$$

$$\tau_{yz} = \vartheta \frac{2G_1 G_2}{G_1 + G_2} x, \tag{35}$$

$$U(x,y) = \frac{G_1 G_2}{G_1 + G_2}\left(R^2 - x^2 - y^2\right), \tag{36}$$

$$V(x,y) = \frac{G_1 G_2}{G_1 + G_2}\left(R^2 - x^2 - y^2\right) + \frac{G_2 x^2 + G_1 y^2}{2}, \tag{37}$$

$$S = \frac{G_1 G_2}{G_1 + G_2} R^4 \pi. \tag{38}$$

The number of collocational points and source points are $N = 6$. The coordinates of the collocational points are (Fig. 2)

$$x_j = R\cos(j-1)\frac{2\pi}{N}, \quad y_j = R\sin(j-1)\frac{2\pi}{N}, \quad (j=1,2,...,6). \tag{39}$$

The coordinates of the source points are (Fig. 2)

$$a_i = (R+r)\cos(i-1)\frac{2\pi}{N}, \quad b_i = (R+r)\sin(i-1)\frac{2\pi}{N}, \quad (i=1,2,...,N). \tag{40}$$

The expression of matrix A_{ij} and vector b_j are as follows

$$A_{ij} = \frac{B_{ij}}{D_{ij}} + \frac{C_{ij}}{D_{ij}}, \quad (i,j=1,2,...,N), \tag{41}$$

$$B_{ij} = 2G_1G_2\cos(j-1)\frac{2\pi}{N}\left[R\cos(j-1)\frac{2\pi}{N} - (r+R)\cos(i-1)\frac{2\pi}{N}\right], \tag{42}$$

$$C_{ij} = 2G_1G_2\sin(j-1)\frac{2\pi}{N}\left[R\sin(j-1)\frac{2\pi}{N} - (r+R)\sin(i-1)\frac{2\pi}{N}\right], \tag{43}$$

$$D_{ij} = G_2\left[R\cos(j-1)\frac{2\pi}{N} - (r+R)\cos(i-1)\frac{2\pi}{N}\right]^2 + G_1\left[R\sin(j-1)\frac{2\pi}{N} - (r+R)\sin(i-1)\frac{2\pi}{N}\right]^2, \tag{44}$$

$$b_j = \frac{G_1 - G_2}{2}R\sin(j-1)\frac{4\pi}{N}, \quad (i,j=1,2,...,6). \tag{45}$$

Following numerical data are used in the numerical example: $R = 0.035$ m, $r = 0.08$ m, $c = R$, $G_1 = 8\times10^{10}$ Pa, $G_2 = 6\times10^{10}$ Pa, $\vartheta = 2\times10^{-3}\ \frac{1}{\text{m}}$, $G_0 = \sqrt{G_1G_2}$. Figure 3 shows the plot of function of approximate solution

$$\omega_a(x,y) = \sum_{i=1}^{6} q_i \ln\left[\frac{G_0}{G_1c^2}(x-a_i)^2 + \frac{G_0}{G_2c^2}(y-b_i)^2\right] \tag{46}$$

and the plot of exact solution $\omega = \omega(x,y)$ given by equation (33) for $x = y$ $0 \le x \le \frac{R}{\sqrt{2}}$. In Fig. 4 and Fig. 5 the exact and approximate solution of shearing stresses $\tau_{xz}(0,y)$ and $\tau_{yz}(x,0)$ are given for $0 \le y \le R$ and for $0 \le x \le R$. The approximate value of torsional rigidity obtained by boundary collocation is

$$S_a = 164189.4053 \text{ Nm}^2. \tag{47}$$

The exact solution of torsional rigidity computed by formula (38) is

$$S = 161634.942 \text{ Nm}^2. \tag{48}$$

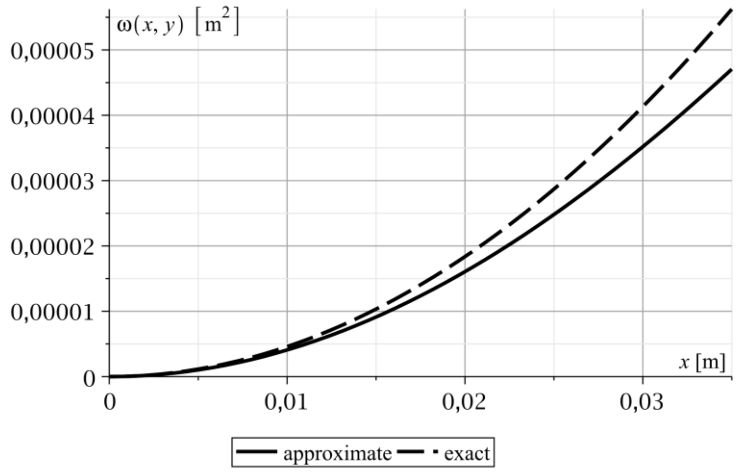

Figure 3. *Plots of the torsion function*

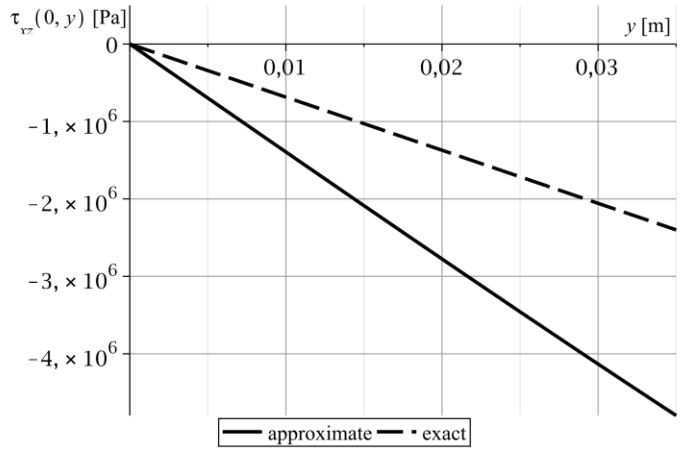

Figure 4. *The graphs of shearing stresses* $\tau_{xz}(0, y)$

3.2. Determination of the Prandtl's stress function and torsional rigidity of bar with isosceles triangle cross section

The boundary curve of the considered bar is bounded with three straight lines as shown in Fig. 6, whose equations are as follows

$$e_1: \ y + mx = 0, \quad e_2: y - mx = 0, \quad e_3: x = H. \tag{49}$$

The number of collocation points and source points are $N = 6$. The position of collocation points and source points is given by the formulae according to Fig. 6: $x_1 = H, \ \ y_1 = 0;$ $x_2 = H, \ \ y_2 = mH;$ $x_3 = \dfrac{H}{2}, \ \ y_3 = \dfrac{H}{2}m;$ $x_4 = 0, \ \ y_4 = 0;$ $x_5 = \dfrac{H}{2}, \ \ y_5 = -\dfrac{H}{2}m;$ $x_6 = H, \ \ y_6 = -mH;$ $a_1 = H + k, \ \ b_1 = 0;$

$a_2 = H + k\cos\alpha,\quad b_2 = Hm + k\sin\alpha;\quad a_3 = \frac{H}{2} - k\sin\alpha,\quad b_2 = \frac{H}{2}m + k\cos\alpha;\quad a_4 = -k,\quad b_4 = 0;$ $a_5 = \frac{H}{2} - k\sin\alpha,\ b_5 = -\frac{H}{2}m - k\cos\alpha;\ a_6 = H + k\cos\alpha,\ b_6 = -Hm - k\sin\alpha.$ Exact analytical solution for the present case exists only if (Lekhnitskii, 1981; Lekhnitskii, 1971)

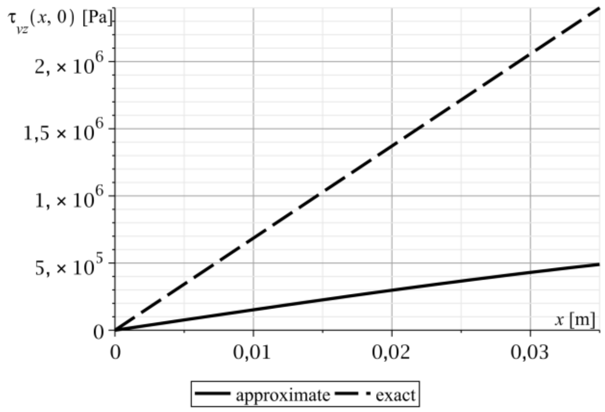

Figure 5. *The graphs of shearing stresses* $\tau_{yz}(x,0)$

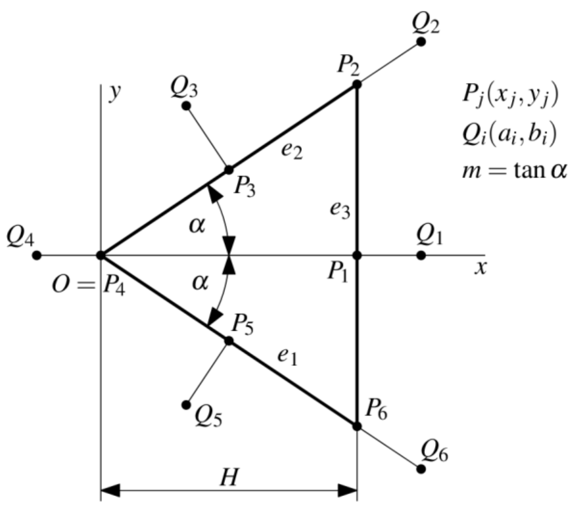

Figure 6. *Orthotropic isosceles triangle cross section*

$$m = \tan\alpha = \sqrt{\frac{G_2}{3G_1}}. \tag{50}$$

It is assumed that m is given by formula (50). In this case the exact solution for the Prandtl's stress function $u = u(x, y)$ and torsional rigidity S are as follows

$$u(x,y)=\frac{3G_1}{2H}\left(y^2-m^2x^2\right)\left(x-H\right), \tag{51}$$

$$S=\frac{\sqrt{3}}{45}\sqrt{G_1G_2}H^4. \tag{52}$$

In the numerical examples the following data are used $G_1=8\times10^{10}$ Pa, $G_2=6\times10^{10}$ Pa, $H=0.06$ m, $m=0.5$, $G_0=\sqrt{G_1G_2}$, $c=H$, $k=0.58$. According to formula (25) the function $V=V(x,y)$ can be represented as

$$V(x,y)=\sum_{i=1}^{N}p_iF_i(x,y). \tag{53}$$

Here,

$$F_i(x,y)=\ln\left[\frac{G_0}{G_1c^2}(x-a_i)^2+\frac{G_0}{G_2c^2}(y-b_i)^2\right],\quad (i=1,2,...,N). \tag{54}$$

The unknown constant p_i $(i=1,2,...,N)$ obtained from a system of linear equations

$$\sum_{i=1}^{N}A_{ij}p_i-b_j=0,\quad (j=1,2,...,N), \tag{55}$$

where

$$A_{ij}=\ln\left[\frac{G_0}{G_1c^2}(x_j-a_i)^2+\frac{G_0}{G_2c^2}(y_j-b_i)^2\right], \tag{56}$$

$$b_j=\frac{G_2x_j^2+G_1y_j^2}{2},\quad (i,j=1,2,...,N). \tag{57}$$

The approximate expression of Prandtl's stress function $U=U(x,y)$ is

$$U(x,y)=-\frac{G_2x^2+G_1y^2}{2}+\sum_{i=1}^{N}p_i\left(\ln\left[\frac{G_0}{G_1c^2}(x-a_i)^2+\frac{G_0}{G_2c^2}(y-b_i)^2\right]\right). \tag{58}$$

The plots of $U=U(x,y)$ and $u=u(x,y)$ are shown in Fig. 7 for $y=0$, $0\le x\le H$. The exact value of torsional rigidity

$$S=34560\ \text{Nm}^2 \tag{59}$$

and the approximate value of torsional rigidity obtained from formula (21)

$$S_a=-\int_{x=0}^{H}\int_{y=-mx}^{mx}\left(x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}\right)\mathrm{d}y\mathrm{d}x=35501.798\ \text{Nm}^2. \tag{60}$$

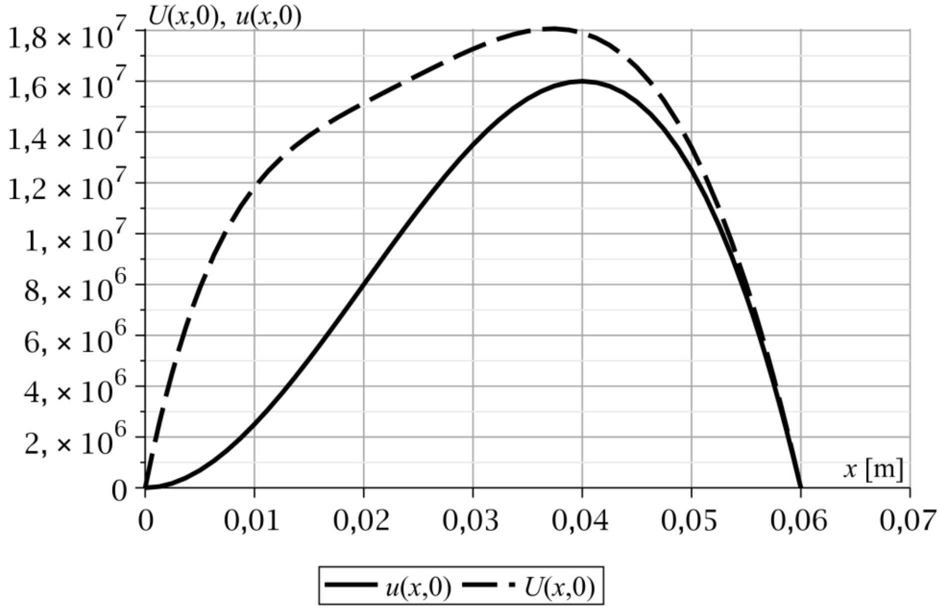

Figure 7. *The plots of functions* $U = U(x,0)$ and $u = u(x,0)$ for $0 \le x \le H$

4. Conclusions

Paper presents an approximate analytical solution for the Saint-Venant torsion of orthotropic bar with solid cross section. The boundary collocation is used to solve the considered Saint-Venant torsion problem. In the applied method the partial differential equations of torsion deformation are satisfied but the boundary conditions for the torsion function and Prandtl's stress function are satisfied only some selected points of the boundary curve of cross section. The formulated method is illustrated by two numerical examples.

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