SAINT-VENANT TORSION OF THIN-WALLED NONHOMOGENEOUS ELLIPTICAL TUBE

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Abstract
This paper deals with the uniform torsion of thin-walled elliptical tube. The material of the tube is nonhomogeneous and it depends on one of the curvilinear coordinates which defines the cross section of thin-walled bar with closed profile. The approximate solution for the stresses, torsion function and torsional rigidity are obtained by the application of two extreme value theorems of linearized elasticity.

Keywords: *Saint-Venant torsion, thin-walled, tube, nonhomogeneous*

1. Introduction

The Saint-Venant torsion of elastic thin-walled bar of closed section is an important topic of textbooks of elasticity (Saada, 1974; Timoshenko and Goodier, 1970; Wang, 1953; Vlasov, 1961; Sun, 2006; Oden, 1967; Renton, 1999; Shama, 2011). The approximate solution of torsion of thin-walled bar with closed profile was given by Bredt (Bredt, 1896). In this paper the approximate solution for thin-walled tube is derived from the Prandtl's formulation for shearing stresses and torsional rigidity. This approach gives a lower bound for the exact value of the torsional rigidity. The torsion function is obtained from the Saint-Venant's theory of uniform torsion. The torsion function formulation yields an upper bound for the torsional rigidity of nonhomogeneous tube. The formulation of the governing equations is provided in orthogonal curvilinear coordinates α and β. *Figure 1* shows the cross section of nonhomogeneous elastic thin-walled tube.

The description of the boundary contours of cross section shown in *Figure 1* is given in curvilinear coordinates α and β which are defined by the following equations *(Figure 1)*

$$x = c\cosh\alpha\cos\beta, \quad 0 < \alpha_1 \le \alpha \le \alpha_2, \quad 0 \le \beta \le 2\pi, \tag{1}$$

$$y = c\sinh\alpha\sin\beta, \quad 0 < \alpha_1 \le \alpha \le \alpha_2, \quad 0 \le \beta \le 2\pi. \tag{2}$$

The coordinate lines of the elliptical coordinate system $O\alpha\beta$ are ellipses and hyperbolas as shown in *Figure 2*.

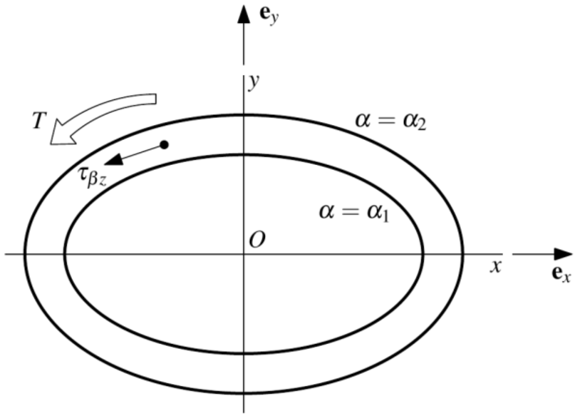

Figure 1. *Thin-walled elliptical closed section*

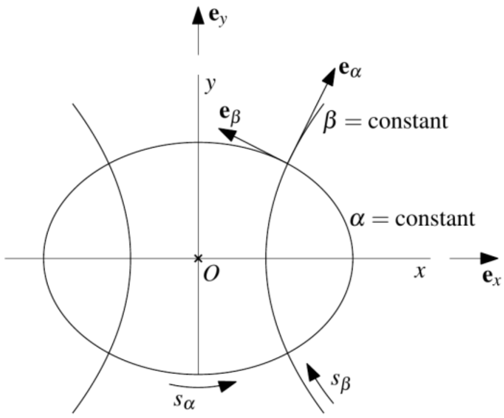

Figure 2. *Coordinate lines of the elliptical coordinate system* $O\alpha\beta$

The unit vectors of $\alpha = \text{constant}$ and $\beta = \text{constant}$ coordinate lines are (Saada, 1974; Lurie, 1970; Renton, 1999)

$$\mathbf{e}_\beta = \frac{1}{h}\left(-\cosh\alpha\sin\beta\mathbf{e}_x + \sinh\alpha\cos\beta\mathbf{e}_y\right), \tag{3}$$

$$\mathbf{e}_\alpha = \frac{1}{h}\left(\sinh\alpha\cos\beta\mathbf{e}_x + \cosh\alpha\sin\beta\mathbf{e}_y\right), \tag{4}$$

where

$$h^2 = \cosh^2\alpha - \cos^2\beta. \tag{5}$$

The expression of the area element $\mathrm{d}A$ in terms of curvilinear coordinates α and β is

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = c^2 h^2 \mathrm{d}\alpha \mathrm{d}\beta. \tag{6}$$

The two-dimensional del operator can be represented as

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y = \frac{1}{ch}\left(\frac{\partial}{\partial \alpha}\mathbf{e}_\alpha + \frac{\partial}{\partial \beta}\mathbf{e}_\beta\right). \tag{7}$$

2. Approximate solution for the Prandtl's stress function

Denote $U = U(\alpha,\beta)$ the Prandtl's stress function of the thin-walled nonhomogeneous tube. According to the Timoshenko variational formulation of Saint-Venant's torsion problem it is known that the Prandtl's stress function is the solution of the following variational problem (Timoshenko and Goodier, 1970; Lurie, 1970; Lebejzon, 1943; Wang, 1953; Slivker, 2007)

$$\Pi_c\left[U(\alpha,\beta)\right] = \max_{\tilde{U}(\alpha,\beta)} \Pi_c\left[\tilde{U}(\alpha,\beta)\right], \tag{8}$$

where

$$\Pi_c\left[\tilde{U}(\alpha,\beta)\right] = 4\int_A \tilde{U}\mathrm{d}A + 4C_1A_1 - \int_A \frac{\left|\nabla\tilde{U}\right|^2}{G(\alpha)}\mathrm{d}A. \tag{9}$$

Here $\tilde{U} = \tilde{U}(\alpha,\beta)$ is a statically admissible stress function which satisfies the boundary condition

$$U(\alpha_2,\beta) = 0, \quad U(\alpha_1,\beta) = C = \text{constant}, \quad 0 \le \beta \le 2\pi. \tag{10}$$

In *equation (9)* $G = G(\alpha)$ is the shear modulus of the tube which may depend on the curvilinear coordinate α and A_1 is the area enclosed by the inner boundary ellipse $(\alpha = \alpha_1)$. The approximate solution of the considered Saint-Venant torsion problem bases on the following assumption: the Prandtl's stress function does not depend on the curvilinear coordinate β, that is

$$\tilde{U}(\alpha,\beta) = U(\alpha), \quad \alpha_1 \le \alpha \le \alpha_2, \quad 0 \le \beta \le 2\pi. \tag{11}$$

Substitution of expression of $\tilde{U} = \tilde{U}(\alpha,\beta)$ given by *equation (11)* provides

$$\begin{aligned}\Pi\left(U(\alpha)\right) = {} & 4\int_0^{2\pi}\int_{\alpha_1}^{\alpha_2} U(\alpha)c^2\left(\cosh^2\alpha - \cos^2\beta\right)\mathrm{d}\alpha\mathrm{d}\beta + \\ & + 2c^2\pi\sinh 2\alpha_1 U(\alpha_1) - \int_0^{2\pi}\int_{\alpha_1}^{\alpha_2}\frac{1}{G(\alpha)}\left(\frac{\partial U}{\partial\alpha}\right)^2\mathrm{d}\alpha\mathrm{d}\beta.\end{aligned} \tag{12}$$

Here, it was used that

$$A_1 = \frac{c^2}{2}\pi\sinh 2\alpha_1. \tag{13}$$

According to the stationary condition of Timoshenko functional (9) we obtain that $U = U(\alpha)$ is the solution of the boundary-value problem formulated by the following equations

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1}{G(\alpha)}\frac{\mathrm{d}U}{\mathrm{d}\alpha}\right) + c^2\cosh 2\alpha = 0, \quad \alpha_1 < \alpha < \alpha_2, \tag{14}$$

$$U(\alpha_2) = 0, \tag{15}$$

$$c^2\sinh\alpha_1 - \frac{1}{G(\alpha_1)}\left(\frac{\mathrm{d}U}{\mathrm{d}\alpha}\right)_{\alpha=\alpha_1} = 0, \tag{16}$$

$$U(\alpha_1) = C = \text{constant}. \tag{17}$$

The solution of the system of *equations (14–17)* are as follows

$$U(\alpha) = \frac{c^2}{2}\int_{\alpha}^{\alpha_2} G(\eta)\sinh 2\eta \mathrm{d}\eta, \tag{18}$$

$$C = U(\alpha_1) = \frac{c^2}{2}\int_{\alpha_1}^{\alpha_2} G(\eta)\sinh 2\eta \mathrm{d}\eta. \tag{19}$$

The approximate value of the torsional rigidity of the thin-walled elliptical tube is obtained from *equation (20)*

$$S = 4\int_A U\mathrm{d}A + 4A_1 U(\alpha_1) - \int_A \frac{1}{G(\alpha)}\left|\nabla U\right|^2 \mathrm{d}A. \tag{20}$$

A detailed computation leads to the next expression

$$S = c^4\int_{\alpha_1}^{\alpha_2}\left[\left(\cosh^2\alpha - \cos^2\beta\right)\int_{\alpha}^{\alpha_2} G(\eta)\sinh 2\eta \mathrm{d}\eta\right]\mathrm{d}\alpha\mathrm{d}\beta + \frac{c^2}{4}\pi\sinh 2\alpha_1\int_{\alpha_1}^{\alpha_2} G(\eta)\sinh 2\eta \mathrm{d}\eta. \tag{21}$$

For given applied torque T the rate of twist ϑ is obtained as

$$\vartheta = \frac{T}{S}. \tag{22}$$

The expression of the shearing stress vector $\boldsymbol{\tau}_z$ in terms of $U = U(\alpha)$ is as follows *(Figure 1)*

$$\boldsymbol{\tau}_z = \tau_{\beta z}\mathbf{e}_\beta = \vartheta\nabla U \times \mathbf{e}_z = \vartheta G(\alpha)\frac{c\sinh\alpha\cosh\alpha}{\sqrt{\cosh^2\alpha - \cos^2\beta}}. \tag{23}$$

3. Approximate solution for the torsion function

It is known the expression of the displacement components of the displacement vector

$$\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + z\mathbf{e}_z \tag{24}$$

can be represented as (Saada, 1974; Timoshenko and Goodier, 1970; Lurie, 1970; Lebejzon, 1943; Wang, 1953)

$$u = -\vartheta yz, \quad v = \vartheta xz, \quad w = \vartheta \omega(x, y). \tag{25}$$

Here $\omega = \omega(x, y)$ denotes the torsion function (Saada, 1974; Wang, 1953; Renton, 1999). The displacement vector can be described in orthogonal curvilinear coordinate system $O\alpha\beta z$ as

$$\mathbf{u} = u_\alpha \mathbf{e}_\alpha + u_\beta \mathbf{e}_\beta + \vartheta\omega(\alpha, \beta)\mathbf{e}_z, \tag{26}$$

where

$$u_\alpha = \vartheta c^2 z \frac{\sin 2\beta}{\sqrt{\cosh^2\alpha - \cos^2\beta}}, \tag{27}$$

$$u_\beta = \vartheta c^2 z \frac{\sinh 2\alpha}{\sqrt{\cosh^2\alpha - \cos^2\beta}}. \tag{28}$$

Application of the strain-displacement of linearized theory of elasticity gives the expressions of shearing strains $\gamma_{\alpha z}$ and $\gamma_{\beta z}$

$$\gamma_{\alpha z} = \frac{\vartheta}{\sqrt{\cosh^2\alpha - \cos^2\beta}} \left[\frac{1}{c} \frac{\partial \omega}{\partial \alpha} + c \sin 2\beta \right], \tag{29}$$

$$\gamma_{\beta z} = \frac{\vartheta}{\sqrt{\cosh^2\alpha - \cos^2\beta}} \left[\frac{1}{c} \frac{\partial \omega}{\partial \beta} + c \sinh 2\alpha \right]. \tag{30}$$

Lagrange's type variational formulation of Saint-Venant's torsion results the following variational problem (Saada, 1974; Lurie, 1970; Slivker, 2007)

$$\Pi_L\left[\omega(\alpha, \beta)\right] = \min_{\tilde{\omega}(\alpha, \beta)} \Pi_L\left[\tilde{\omega}(\alpha, \beta)\right], \tag{31}$$

where

$$\Pi_L\left[\tilde{\omega}(\alpha, \beta)\right] = \int_A G(\alpha)\left[\tilde{\gamma}_{\alpha z}^2 + \gamma_{\beta z}^2\right] \mathrm{d}A. \tag{32}$$

In *equation (32)* $\tilde{\omega} = \tilde{\omega}(\alpha, \beta)$ is an arbitrary kinematically admissible torsion function (Saada, 1974; Lurie, 1970; Slivker, 2007), $\tilde{\gamma}_{\alpha z}$ and $\gamma_{\beta z}$ are obtained from formulae (29) and (30). By the use of kinematically admissible function

$$\tilde{\omega}(\alpha, \beta) = K \sinh 2\alpha \sin 2\beta \tag{33}$$

we get

$$\Pi_L\left[\tilde{\omega}(\alpha, \beta)\right] = \Pi_L(K) = k_0 + k_1 K + k_2 K^2, \tag{34}$$

that is

$$\Pi_L(K) = k_2\left(K^2 + k_1K + \frac{k_1^2}{4k_2^2}\right) + k_0 - \frac{k_1^2}{4k_2} = k_2\left(K + \frac{k_1}{2k_2}\right)^2 + k_0 - \frac{k_1^2}{4k_2}. \tag{35}$$

Here,

$$k_0 = c^4 \int_0^{2\pi}\int_{\alpha_1}^{\alpha_2} G(\alpha)\left(\sinh^2 2\alpha + \sin^2 2\beta\right)\mathrm{d}\alpha\mathrm{d}\beta, \tag{36}$$

$$k_1 = 2c^2 \int_0^{2\pi}\int_{\alpha_1}^{\alpha_2} G(\alpha)\left(\sin^2 2\beta \cosh 2\alpha + \sinh^2 2\alpha \cos 2\beta\right)\mathrm{d}\alpha\mathrm{d}\beta, \tag{37}$$

$$k_2 = 4 \int_0^{2\pi}\int_{\alpha_1}^{\alpha_2} G(\alpha)\left(\cosh^2 2\alpha \sin^2 2\beta + \sinh^2 2\alpha \cos^2 2\beta\right)\mathrm{d}\alpha\mathrm{d}\beta. \tag{38}$$

From *equation (35)* it follows that the sharpest upper bound in this case

$$S_U = k_0 - \frac{k_1^2}{4k_2}, \quad K = -\frac{k_1}{2k_2}. \tag{39}$$

4. Bredt's type approximate solution

Bredt's formulation neglects the dependence of shearing stresses $\tau_{\beta z}$ on α. From this assumption it follows that

$$\tau_{\beta z}(\beta) = \frac{T}{2A_0 t(\beta)}, \tag{40}$$

where t is the thickness of thin-walled tube

$$t(\beta) = t_1(\beta) + t_2(\beta), \quad 0 \le \beta \le 2\pi, \tag{41}$$

$$t_i(\beta) = c\sqrt{\cos^2\beta\left(\cosh\alpha_i - \cosh\alpha_0\right)^2\left(\sinh\alpha_i - \sinh\alpha_0\right)}, \quad (i = 1, 2), \tag{42}$$

$$\alpha_0 = \frac{1}{2}\left(\alpha_1 + \alpha_2\right), \quad A_0 = \frac{c^2}{2}\pi\sinh 2\alpha_0. \tag{43}$$

The mean value of shear modulus in thickness direction is

$$G_m = \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} G(\alpha)\mathrm{d}\alpha. \tag{44}$$

Bredt proved that approximate value of torsional rigidity of thin-walled beam with closed profile is

$$S_B = G_m \frac{4A_0^2}{\oint \frac{\mathrm{d}s}{t(s)}}. \tag{45}$$

In the present problem this formula yields the result

$$S_B = G_m \frac{c^4 \pi \sinh^2 2\alpha_0}{\int_0^{2\pi} \frac{c\sqrt{\cosh^2 \alpha_0 - \cos^2 \beta}}{t(\beta)} \mathrm{d}\beta}. \tag{46}$$

From the formula (23) it follows that

$$\tau_\beta(\alpha, \beta) = \frac{T}{S} G(\alpha) \frac{c \sinh \alpha \cosh \alpha}{2\sqrt{\cosh^2 \alpha - \cos^2 \beta}}. \tag{47}$$

Comparison of *equation (40)* with *(47)* gives

$$\frac{G(\alpha_0)}{2S} \frac{\sinh \alpha_0 \cosh \alpha_0}{\sqrt{\cosh^2 \alpha_0 - \cos^2 \beta}} = \frac{1}{c^2 \pi \sinh 2\alpha_0 t(\beta)}. \tag{48}$$

In *equation (48)* an average value of the thickness is used instead of

$$\frac{t(\beta)}{\sqrt{\cosh^2 \alpha_0 - \cos^2 \beta}} \approx \frac{1}{2\pi} \int_0^{2\pi} \frac{t(\beta)}{\sqrt{\cosh^2 \alpha_0 - \cos^2 \beta}} \mathrm{d}\beta. \tag{49}$$

Introducing this approximation into formula (48) gives a new formula to the torsional rigidity

$$\tilde{S}_B = \frac{c^3}{4} G(\alpha_0) \sinh^2 2\alpha_0 \int_0^{2\pi} \frac{t(\beta)}{\sqrt{\cosh^2 \alpha_0 - \cos^2 \beta}} \mathrm{d}\beta. \tag{50}$$

5. Numerical example

In the numerical example the following data are applied $c = 10$ m, $\alpha_1 = 0.48$, $\alpha_2 = 0.5$, $T = 10$ kNm, $G_0 = 2 \times 10^8$ Pa, $\lambda = 0.1$, $G(\alpha) = G_0 \exp(\lambda\alpha)$. The explicit form of the Prandtl's stress function is as follows

$$U(\alpha, \lambda) = \frac{G_0}{4} c^2 \left(\cosh 2\alpha_2 - \cosh 2\alpha\right) \exp(\lambda\alpha), \quad \alpha_1 \le \alpha \le \alpha_2, \quad 0 \le \varphi \le 2\pi. \tag{51}$$

Figure 3 shows the dependence of stress function from the material parameter λ. Let

$$r(\alpha, \lambda) = \frac{G_0}{8} c^4 \pi \left(\lambda \cosh 2\alpha - \lambda \cosh 2\alpha_2 + 2\sinh 2\alpha\right)^2 \exp(\lambda\alpha) \tag{52}$$

be. The torsional rigidity can be expressed as

$$S(\lambda) = \int_{\alpha_1}^{\alpha_2} r(\alpha, \lambda) \mathrm{d}\alpha. \tag{53}$$

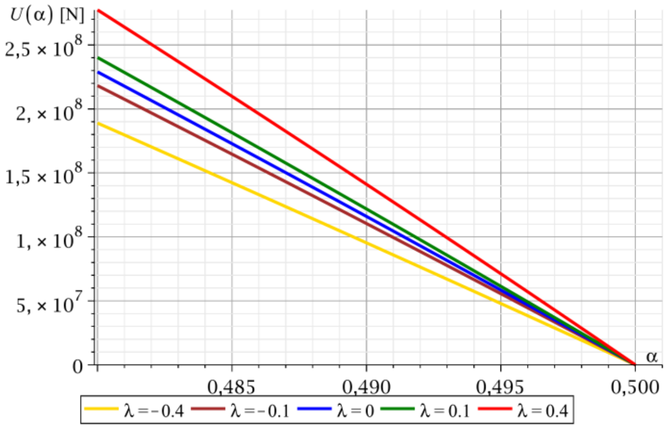

Figure 3. *Plots of Prandtl's stress function for different value of* λ

The dependence of torsional rigidity on the geometrical parameter $p = \alpha_2 / \alpha_1$ is shown in *Figure 4* for $1.02 \le p \le 1.1$. *Figure 5* shows the graphs of shearing stress $\tau_{\beta z}(\alpha, \beta)$ for five different values of β $\left(\beta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right)$. The comparison of shearing stresses obtained from *equation (23)* and Bredt's formula (40) is shown in *Figure 6*. Bredt's formula (40) gives

$$\tau_0(\beta) = \frac{T}{c^2 \pi \sinh(2\alpha_0) t(\beta)}, \tag{54}$$

and application of *equation (23)* for $\alpha = \alpha_0 = 0.5(\alpha_1 + \alpha_2)$ yields the result

$$\tau(\alpha_0, \beta) = \frac{Tc}{2S\sqrt{\cosh^2 \alpha_0 - \cos^2 \beta}} G(\alpha_0) \sinh 2\alpha_0. \tag{55}$$

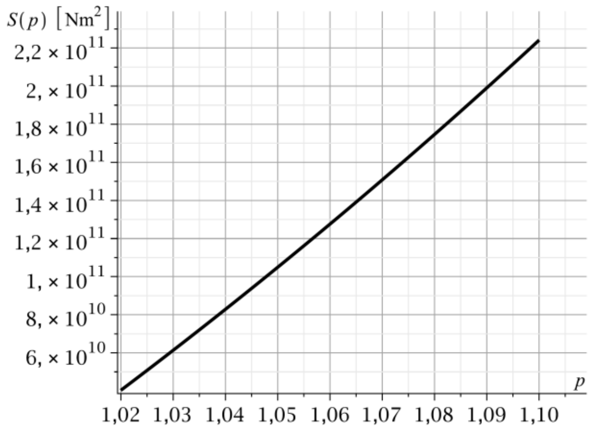

Figure 4. *The variation of the torsional rigidity as a function of* $p = \alpha_2 / \alpha_1$ *for* $\lambda = 0.1$

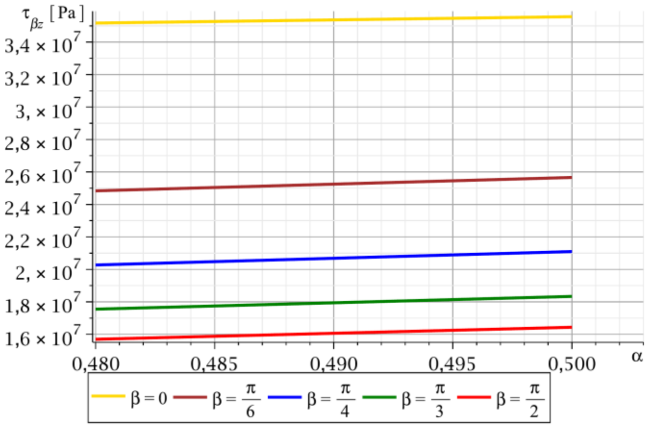

Figure 5. *The graphs of the shearing stresses* $\tau_{\beta z}$ *for the five different values of* β $(\lambda = 0.1)$

It should be noted that in the developed numerical example the graphs of $\tau_0(\beta)$ and $\tau\left(\alpha_0, \beta\right)$ are practically the same *(Figure 6)*. In the present problem the following values obtained for the torsional rigidity S:

a) *Equation (21)* gives a lower bound for S which is

$$S_L = 8.647981022 \times 10^{10} \text{ Nm}^2. \tag{56}$$

b) From *equation (39)* an upper bound can be derived for S

$$S_U = 1.01548926 \times 10^{11} \text{ Nm}^2. \tag{57}$$

c) Application of Bredt's formula (46) yields

$$S_B = 5.012122612 \times 10^{10} \text{ Nm}^2. \tag{58}$$

d) Bredt's type formula (50) provides the result

$$\tilde{S}_B = 8.644820555 \times 10^{10} \text{ Nm}^2. \tag{59}$$

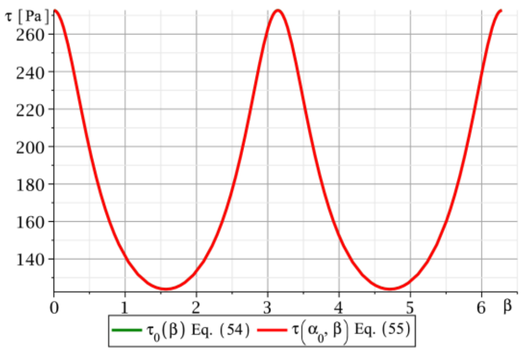

Figure 6. *Comparison of the shearing stresses obtained from Prandtl's solution and Bredt's formula*

Based on the above results it is advisable to use formulae (21) and (50) from which come closed results to each other to calculate the torsional rigidity. Note that in this numerical example the numerical value S_U suits that the Prandtl's formulation indeed gives a lower bound on the torsional stiffness.

6. Conclusions

In this paper Saint-Venant's torsion of thin-walled elastic tube is considered. The formulation of the problem in a curvilinear coordinate system is given. Paper presents approximate expressions for Prandtl's stress function, shearing stresses and torsional rigidity. The derived results based on Saint-Venant's theory of uniform torsion are compared with the results obtained by the application of Bredt's theory.

References

[1] Saada, A. S. (1974). *Elasticity: Theory and applications in engineering.* Pergamon Press, New York.
[2] Timoshenko, S. P., Goodier, J. N. (1970). *Theory of elasticity.* McGraw-Hill, New York.
[3] Lurie, A. I. (1970). *Theory of elasticity.* Nauka, Moscow (in Russian).
[4] Lebejzon, L. S. (1943). *Solution of problems of elasticity by variational methods.* Gosztehizdat, Moscow (in Russian).
[5] Wang, C. T. (1953). *Applied elasticity.* McGraw-Hill, New York.
[6] Vlasov, V. Z. (1961). *Thin-walled elastic beams.* English translation: National Science Foundation, Oldbourne Press, London.
[7] Sun, C. T. (2006). *Mechanics of aircraft structures.* (Second edition), John Wiley and Sons Inc. New Jersey.
[8] Oden, J. T. (1967). *Mechanics of elastic structures.* McGraw-Hill.
[9] Renton, J. D. (1999). *Elastic beams and frames.* Camford Books, Oxford.
[10] Slivker, V. (2007). *Mechanics of structural elements: Theory and Applications.* Springer-Verlag, Berlin. **https://doi.org/10.1007/978-3-540-44721-4**
[11] Shama, M. (2011). *Torsion and shear stresses in ships.* Springer, New York. **https://doi.org/10.1007/978-3-642-14633-6**
[12] Bredt, R. (1896). Kritische Bemerkungen zur Drehungselastizität. *Zeitschrift des Vereines Deutscher Ingenieure*, 40 (28), pp. 785–790.