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EXISTENCE AND UNIQUENESS THEOREMS FOR PEXIDER ADDITIVE EQUATIONS

Tamás Glavosits
assistant professor, University of Miskolc, Institute of Mathematics, Department of Applied Mathematics
3515 Miskolc, Miskolc-Egyetemváros, e-mail: tamas.glavosits@uni-miskolc.hu

Zsolt Karácsony
associate professor, University of Miskolc, Institute of Mathematics, Department of Applied Mathematics
3515 Miskolc, Miskolc-Egyetemváros, e-mail: zsolt.karacsony@uni-miskolc.hu

Abstract
The aim of this presentation is to show a generalization of the well-known Rimán's Extension Theorem and the main steps of the way leading to it.

Keywords: *Interval arithmetic, Restricted Pexider-Additive functional equations, Archimedean ordered groups*

1. Introduction

In this presentation restricted Pexider additive functional equations are investigated. If $X = X(+)$, $Y = Y(+)$ are algebraic structures, $F : X \to Y$ is a function such that

$$F(x+y) = F(x) + F(y) \qquad (x, y \in X),$$

then function F is said to be additive. If $X = X(+)$, $Y = Y(+)$ are algebraic structures, $D \subseteq X^2 := X \times X$ is a set,

$$Dx := \{u \in X \mid \exists (v \in X) : (u, v) \in D\},$$

$$Dy := \{v \in X \mid \exists (u \in X) : (u, v) \in D\},$$

$$Dx+y := \{z \in X \mid \exists ((u, v) \in D) : z = x+y\},$$

$F : Dx+y \to Y$, $G : Dx \to Y$, $H : Dy \to Y$ are unknown functions such that

$$F(x+y) = G(x) + H(y) \qquad (x, y) \in D)$$

then this equation is said to be restricted Pexider additive functional equation.

The main purpose of this presentation is to give the general solution of the restricted Pexider-additive functional equation in the case that $X(+,\leq)$ is an Archimedean ordered Abelian group, D is an open subset of X^2, $Y(+)$ is an Abelian group. This result is a generalisation of the well-known Rimán's Extension Theorem (Aczél,1966).

Knowledge of the sums and products of intervals is required for our extension and uniqueness theorems.

This presentation is structured as follows:

In section 2, as a preliminary knowledge of our main results, the sums an products of open intervals in ordered semi-groups is investigated. A fact on number theory is also mentioned in this section. This section is based on the paper (Glavosits and Karácsony, 2021b).

In section 3, as the most important preliminary knowledge for our main results, the sums and products of intervals of Archimedean ordered groups and fields is summarized. This section is based on the paper (Glavosits and Karácsony 2021a).

In section 4 the local version of the Extension Theorem for additive functions is shown. This section is based on the paper (Glavosits and Karácsony, 2021a).

Finally, in section 5, our main result, the generalization of the Rimán's Theorem can be found. This is the 'global' version of the Theorems can be found in section 4. This section is based on the paper (Glavosits, in preparation).

Table 1.

x	1	$\boxed{2}$	$\boxed{3}$	4	$\boxed{5}$	6	$\boxed{7}$	8	9	10	$\boxed{11}$	12	$\boxed{13}$
$g(x)$	1	$\boxed{2}$	$\boxed{4}$	4	$\boxed{6}$	6	$\boxed{10}$	10	10	10	$\boxed{12}$	12	$\boxed{16}$

2. Sum of intervals in ordered semi-group

2.1. A sufficient condition for additivity and homogeneity properties of interval

Definition 2.1. Let $S = S(+,\leq)$ be a partially ordered Abelian semigroup. Consider the following properties:

1. $S = S(+)$ is cancellative in the sense that $x+ z = y + z$ implies $x = y$ for all $x, y, z \in S$.
2. If $x < y$ then there exists an element $z \in S$ such that $y = x+ z$ for all $x, y, z \in S$.
3. $x \neq x+ y$ for all $x, y \in S$.
4. The strict order $<$ is co-directed in the sense that for all $x, y \in S$ there exists an element $z \in S$ such that $z < x$ and $z < y$.

Theorem 2.2. Let $S = S(+,\leq)$ be an ordered semi-group with properties of Definitions 2.1. Let $\alpha, \beta, \gamma \in S$ such that $\alpha < \beta$. Then

$$\gamma +]\alpha, \beta[=]\gamma + \alpha, \gamma + \beta[\quad \text{and} \quad \gamma +]\alpha, \beta] =]\gamma + \alpha, \gamma + \beta]$$

where $]x, y] := \{z \in S | x < z \text{ and } z \leq y\}$ for all $x, y \in S$.

Theorem 2.3. Let $S = S(+,\leq)$ be an ordered semi-group with properties of Definition 2.1. Let $a, b, c, d \in S$ such that $a < b$ and $c < d$. Then

$$]a, b[+]c, d[=]a+ c, b+ d[.$$

2.2. On an application with number theory background

It is well-known that the sum of nonempty open interval of the ring of all integers is also an interval.

It is also well-known that the product of nonempty open interval of the ordered ring Z is not always an open interval. Define the set $I_x \subseteq Z_+$ by

$$I_x := \{1, 2, \ldots, x\} \qquad (x \in Z_+).$$

The set Ix is an open interval of Z, but Ix·Ix is not always, for example $I_3 \cdot I_3 = \{1, 2, 3, 4, *, 6, *, *, 9\}$.

Define the function $g : Z_+ \to Z_+$ by

$$g(x) := \max \{y \in Z+ \mid Iy \subseteq Ix \cdot Ix\} \qquad (x \in Z_+).$$

It is easy to see that for example $g(3) = 4$. Now we give a table which contains the value of x and g(x) for some small integer x. The above table suggests the following Theorem.

Theorem 2.4. The function g has the following properties:

1. the function g is increasing;
2. $g(x-1) < g(x)$ if and only if x is prime;
3. $g(p_n) = p_{n+1} - 1$ where $p_1, p_2, \ldots$ is the increasing sequence of all prime numbers.

The proof of the above Theorem is based on the the Bertrand's postulate which states that there exists a prime number in the interval $[n, 2n]$ for all $n \in Z_+$. This postulate was proved for the first time by P. L. Chebishev in 1850 and simplified later by P. Erdős in 1932 (Erdős, 1932) due to M. El Bachraoui (El Bachraoui, 2006).

2.3. Additional examples and open problems

Example 2.5. Define the set K_1 by

$$K_1 := \{a + b\sqrt{2} \mid a \in Q_+, b \in Q_+\}.$$

Then $K_1 = K_1(+,\leq)$ and $K_1 = K_1(\cdot,\leq)$ are ordered semi-groups where $+$, $\cdot$ and $\leq$ are the usual addition, multiplication and order in the real line.

K_1 has very interesting properties. Let $a, b, c, d \in K_1$ such that $a < b$ and $c < d$.

- $]a, b[+]c, d[=]a+ c, b+ d[$, but $c +]a, b[\neq]c+ a, c+ b[$, in the sense, that the equality does not always satisfied.
- $]a, b[\cdot]c, d[=]ac, bd[$, but $c \cdot]a, b[\neq]ca, cb[$, in the sense, that the equality does not always satisfied.

Definition 2.6. An ordered semi-group $X(\leq)$ is said to be dense if for all $x, y \in X$ with $x < y$ there exists an $z \in X$ such that $x < z < y$.

Conjecture 2.7. If $S = S(+,\leq)$ is a dense ordered semi-group, then

$$]a, b[+]c, d[=]a+ c, b+ d[$$

for all $a, b, c, d \in S$ such that $a < b$, $c < d$.

Definition 2.8. An ordered monoid $S(+,\leq)$ (with identity element 0) is said to be Archimedean ordered, if for all $0 < x$ and $0 < y$ there exists a positive integer n such that

$$x < ny := y + \cdots + y.$$

3. Sums and products in ordered groups and fields

The foundations of the so-called interval arithmetic were laid by E. Moore, the first appearance of this topic was in 1959 (Moore, 1959), see also (Moore, 1962; Moore, 1966) and (Cloud et al., 2009). Now, we shall show the Moore's formulas

$$[\underline{a},\overline{a}] + [\underline{b},\overline{b}] = [\underline{a}+\underline{b},\overline{a}+\overline{b}]$$
$$[\underline{a},\overline{a}] - [\underline{b},\overline{b}] = [\underline{a}-\overline{b},\overline{a}-\underline{b}]$$
$$[\underline{a},\overline{a}] \cdot [\underline{b},\overline{b}] = [\min(\underline{a}\underline{b},\underline{a}\overline{b},\overline{a}\underline{b},\overline{a}\overline{b}),\max(\underline{a}\underline{b},\underline{a}\overline{b},\overline{a}\underline{b},\overline{a}\overline{b})\ ]$$
$$\frac{[\underline{a},\overline{a}]}{[\underline{b},\overline{b}]} = [\underline{a},\overline{a}] \cdot \left[\frac{1}{\overline{b}},\frac{1}{\underline{b}}\right] \qquad 0 \notin [\underline{b},\overline{b}]$$

for all $\underline{a}, \overline{a}, \underline{b}, \overline{b} \in$ R with $\underline{a} \leq \overline{a}$ and $\underline{b} \leq \overline{b}$.

Due to (Jaulin et al., 2001) the results of Moore was extended to open ended unbounded intervals by R. J. Hanson (1968) (Hanson, 1968), W. Kahan (Kahan, 1968), E. Davis (Davis, 1987).

The famous Kohan-Novoa-Ratz arithmetic concerning to the division by an interval containing zero can be found in (Kearfott, 1996).

3.1. The sums of open ended bounded intervals in ordered groups

To obtained results will be used to extend additive functions (Glavosits, Karácsony, it will appear soon).

Theorem 3.1. If G = G(+,≤) is an ordered dense Abelian group, $\underline{a}, \overline{a}, \underline{b}, \overline{b} \in$ G with $\underline{a} \leq \overline{a}$ and $\underline{b} \leq \overline{b}$, then

$$[\underline{a},\overline{a}] + [\underline{b},\overline{b}] = [\underline{a}+\underline{b},\overline{a}+\overline{b}].$$

Theorem 3.2. If G = G(+, ·,≤) is an Archimedean ordered group, then the following assertions are equivalent:

1. G is dense
2. $[\underline{a},\overline{a}] + [\underline{b},\overline{b}] = [\underline{a}+\underline{b},\overline{a}+\overline{b}]$ for all $\underline{a}, \overline{a}, \underline{b}, \overline{b} \in$ G with $\underline{a} \leq \overline{a}$ and $\underline{b} \leq \overline{b}$.
3. G(+,≤) is not isomorphic to the ordered group Z = Z(+,≤) (which is the group of all integers).

3.2. The products of interval in ordered fields

In this subsection we investigate the products of open ended bounded intervals in ordered fields F =F(+, ·,≤). Let $\underline{a}, \overline{a}, \underline{b}, \overline{b} \in$ F with $\underline{a} \leq \overline{a}$ and $\underline{b} \leq \overline{b}$. Define the intervals a and b by

a :=] $\underline{a}, \overline{a}$ [and b :=] $\underline{b}, \overline{b}$ [.

As a temporary device, use the notation for any open ended bounded interval x that

0 < x, if 0 < x for all x ∈ x,

x < 0, if x < 0 for all x ∈ x.

Theorem 3.3. If F = F(+, ·,≤) is an ordered field, $\underline{a}, \overline{a}, \underline{b}, \overline{b} \in$ F with $\underline{a} \leq \overline{a}$ and $\underline{b} \leq \overline{b}$, 0 < a and 0 < b, then a · b =]$\underline{a}\underline{b}, \overline{a}\overline{b}$ [.

First, we investigate the case, when the point 0 is an interior point neither of the interval a nor of the interval b.

Theorem 3.4. If F = F(+, ·,≤) is an ordered field, then

1. If 0 < a and 0 < b, then a · b =]$\underline{ab}, \overline{ab}$[.
2. If a < 0 and b < 0, then a · b =]$\overline{ab}, \underline{ab}$[.
3. If a < 0 and 0 < b, then a · b =]$\underline{a}\overline{b}, \overline{a}\underline{b}$[.
4. If b < 0 and 0 < a, then a · b =] $\overline{a}\underline{b}, \underline{a}\,\overline{b}$[.

Now we investigate the case, when the point 0 is an interior point either of the interval a or of the b.

Theorem 3.5. If F = F(+, ·,≤) is an ordered field, then

1. If 0 ∈ a and 0 < b, then a · b =]$\underline{ab}, \overline{ab}$[.
2. If 0 ∈ a and b < 0, then a · b =] $\overline{ab}, \underline{ab}$[.
3. If 0 ∈ b and 0 < a, then a · b =]$\underline{a}\overline{b}, \overline{a}\underline{b}$[.
4. If 0 ∈ b and a < 0, then a · b =]$\overline{a}\underline{b}, \underline{a}\,\overline{b}$[.

Finally, we investigate the case, when the point 0 is an interior point both of] $\underline{a}, \overline{a}$ [and] $\underline{b}, \overline{b}$ [.

Theorem 3.6. If F = F(+, ·,≤) is an ordered field, then If 0 ∈ a∩b, then

$$a \cdot b =]\min\{\underline{a}\overline{b}, \overline{a}\underline{b}\}, \max\{\underline{ab}, \overline{ab}\}[.$$

4. Existence and uniqueness theorems (local version)

The restricted additive functional equations have been previously studied by many researchers. In the book (Graham et al., 2013) Part IV. Geometry, Section Extension of Functional Equations p. 447–460 the authors cite numerous papers that investigate the cases when there exists an additive function F:R → R such that the function F extends the function f . An incomplete list of such papers is given bellow:

In the paper (Aczél and Erdős, 1965) D = (D_+ ∪ {0})^2.

In the book (Aczél, 1966) the first appearance of the concept of quasi extension can be found. An additive function a is said to be quasi extension of the function f if f is additive on a set D ⊆ R^2 and there exists an additive function a:R→ R and exist constants c_1, c_2 ∈ R such that

$$f(u) = a(u) + c_1 \qquad (u \in Dx);$$

$$f(v) = a(v) + c_2 \qquad (v \in Dy);$$

$$f(z) = a(z) + c_1 + c_2 \qquad (z \in Dx{+}y).$$

In the paper (Daróczy and Lononczi, 1967) the cases D = R_+^2 and D is an open interval of the real line containing the origin is investigated. In this paper the notations Dx, Dy, Dx+y has appeared first.

In (Székelyhidi, 1972) the author generalizes the above result that D ⊆ R^2 is an arbitrary open set, D_0 = Dx ∪Dy ∪ Dx+y, f : D_0 → R is a function such that

$$f(x+ y) = f(x) + f(y) \qquad (x, y) \in D.$$

In (Rimán, 1976) a simple extension theorem can be found for Pexider additive functional equation where the additivity is fulfilled in a nonempty connected open set of R^2.

In the article (Aczél, 1983) D = H(I) where I is a nonempty open interval of the real line and the set H(I) is defined by

$$H(I) :=\{(x, y) \in R^2 \mid x, y, x+ y \in I\}.$$

The set H(I) is a hexagon, sometimes a triangle or the emptyset.

In the book (Kuczma, 1985) $D \subseteq R^N$ is a nonempty connected open set. The extension is brought back to the theory of the midkonvex functions, but in this book the author does not consider the restricted Pexider additive functional equations.

In the article (Radó and Baker, 1987) an extension theorem can be found for restricted Pexider additive functional equations where $D \subseteq R^N$ is a nonempty connected open set.

In the book (Aczél and Thombres, 1989) several functional equations are considered in more general abstract algebraic settings.

4.1. The Euclidean division in Archimedean ordered groups

Euclid's Elements (Euclid and Fitzpatrick, 2007) is one of the most influential mathematical textbook written more than two thousand years ago. In this textbook (book X, proposition 3.) there is an algorithm using the so-called Euclidean or remainder division to give the greatest common measure of two given commensurable magnitudes. We use the modern version of this Theorem to give our Extension Theorem for restricted additive functional equations.

Theorem 4.1. If $G = G(+,\leq)$ is an Archimedean ordered group, $x, y \in G$ with $y \neq 0$, then there uniquely exists an integer q and an element $r \in G$ such that

$$x = qy + r \quad \text{where} \quad 0 \leq r < |y|.$$

4.2. Extension Theorem for additive functional equations

Theorem 4.2. Let $G(+,\leq)$ be an Archimedean ordered dense Abelian group, Y (+) be a group, $\varepsilon \in G_+$ and $f :]-2\varepsilon, 2\varepsilon[\to Y$ be a function such that

$$f(x+ y) = f(x) + f(y) \quad (x, y \in]-\varepsilon, \varepsilon[),$$

then there exists an additive function $a : G\to Y$ which extends the function f.

5. Uniqueness Theorem for additive functional equations

Theorem 5.1. Let $G = G(+,\leq)$ be an Archimedean ordered Abelian group, Y (+) be a group, and $a : G\to Y$ be an additive function. If there exist constants $\alpha, \beta \in G$ with $\alpha < \beta$ and $c \in Y$ such that

$$a(x) = c \qquad (x \in]\alpha, \beta[),$$

then $a(x) = 0$ for all $x \in G$.

Corollary 5.2. Let $G = G(+,\leq)$ be an Archimedean ordered Abelian group, Y (+) be a group, and $a_1, a_2 : G \to Y$ be additive functions. If there exists a nonempty open interval $]\alpha, \beta[\subseteq X$ and a constant $c \in Y$ such that

$$a_1(x) = a_2(x) + c \qquad (x \in]\alpha, \beta[),$$

then $a_1(x) = a_2(x) = 0$ for all $x \in G$.

5.1. Extension Theorem for Pexider additive functional equation 'local version'

If $G(+,\leq)$ is an ordered group, then the set $B(x, \varepsilon)$ is defined by

$$B(x,\varepsilon) := \begin{cases}]x-\varepsilon, x+\varepsilon[\,, \text{if } x \in G; \\ ]x_1-\varepsilon, x_1+\varepsilon[\times]x_2-\varepsilon, x_2+\varepsilon[\,, \text{if } x \in G^2. \end{cases}$$

which is the neighbourhood of a point $x \in G$ or $x := (x_1, x_2) \in G\times G$ with radius $\varepsilon \in G_+$.

Theorem 5.3. If $G(+,\leq)$ is an Archimedean ordered, dense, Abelian group, $Y(+)$ is a group, $x_0, y_0 \in G$, $\varepsilon \in G_+$, and $f : B(x_0 + y_0, 2\varepsilon)\to Y$, $g : B(x_0, \varepsilon)\to Y$, $h : B(y_0, \varepsilon)\to Y$ are functions such that

$$f(x+y) = g(x) + h(y) \qquad ((x, y) \in B((x_0, y_0), \varepsilon)),$$

then there exists an additive function $a : G\to Y$ and exist constants $c, d \in Y$ such that

$$f(u) = a(u) + c + d \quad (u \in B(x_0 + y_0, 2\varepsilon)),$$

$$g(v) = a(v) + c \quad (v \in B(x_0, \varepsilon)),$$

$$h(z) = a(z) + d \quad (z \in B(y_0, \varepsilon)).$$

5.2. Topology generated by the open intervals of an Archimedean ordered Abelian groups

Let $G = G(+,\leq)$ be an ordered group, $X \in \{G,G^2\}$ and $D \subseteq X$. The set D is said to be open if, for every point x in D, there exists $\varepsilon \in G_+$ such that $B(x, \varepsilon) \subseteq D$. A subset $D \subseteq X$ is said to be well-chained, if for all
$x, y \in D$, there exists a finite sequence $B_i := B(x_i, \varepsilon_i)$ $(i = 0, 1, \ldots, n)$ such that

1. $B_i \subseteq D$ for all $i = 0, 1, \ldots, n$.
2. $x \in B_0$, $y \in B_n$.
3. $B_{i-1} \cap B_i \neq \emptyset$ for all $i = 1, \ldots, n$.

A subset C of a nonempty, open set $D \subseteq X$ is a component of D if C is a maximal well-chained, open subset of D with respect the inclusion.

A topological space $X(T)$ is said to be separable, if there exists a subset $Y \subseteq X$ which is countable, infinite, and dense (in X).

Theorem 5.4. If $G = G(+,\leq)$ is an ordered group, $X \in \{G,G^2\}$ and $D \subseteq X$ is a nonempty, well-chained, open set, then

1. D is a disjoint union of its components;
2. If X is separable, then D has countable components.

5.3. Extension Theorem for Pexider additive functional equation 'global version'

Now we can easily give the generalisation of the well-known Rimán's Extension Theorem.

Theorem 5.5. Let $G(+,\leq)$ be an Archimedean ordered, dense, Abelian group, $D \subseteq G^2$ be an open set with components $\{D_i \mid i \in I\}$, and Y be an Abelian group. The functions $f : D_{x+y} \to Y$, $g : D_x \to Y$, $h : D_y \to Y$ are solutions of the functional equation

$$f(x+y) = g(x) + h(y) \qquad ((x, y) \in D)$$

if and only if there exists a family of additive functions ai : $G \to Y$ $(i \in I)$ and exist families of constants $c_i, d_i \in Y$ $(i \in I)$ such that

$$
\begin{array}{ll}
f(u) = a_i(u) + c_i + d_i & (u \in (D_i)x+y), \\
g(v) = a_i(v) + c_i & (v \in (D_i)x), \\
h(z) = a_i(z) + d_i & (z \in (D_i)y,)
\end{array}
$$

with the properties:

1. If $(D_i)_{x+y} \cap (D_j)_{x+y} \neq \emptyset$, then $a_i = a_j$ and $c_i + d_i = c_j + d_j$.
2. If $(D_i)x \cap (D_j)x \neq \emptyset$, then $a_i = a_j$ and $c_i = c_j$.
3. If $(D_i)y \cap (D_j)y \neq \emptyset$, then $a_i = aj$ and $d_i = d_j$.

for all $i, j \in I$ $i \neq j$.

Open Problem 5.6. Let $S(+,\leq)$ be an Archimedean ordered dense Abelian monoid, Y (+) be an Abelian semi-group, $a, b, c, d \in S$ such that $a < b$ and $c < d$, and

$$R :=]a, b[\times]c, d[.$$

If the functions $f : Rx+y \to Y$, $g :]a, b[\to Y$, $h :]c, d[\to Y$, satisfy the equation

$$f(x+ y) = g(x) + h(y) \qquad ((x, y) \in R),$$

then there exists an additive function $a : S \to Y$ and exist constants $\alpha, \beta \in Y$ such that

$$
\begin{array}{ll}
f(u) = a(u) + \alpha + \beta & (u \in Rx+y), \\
g(v) = a(v) + \alpha & (v \in]a, b[), \\
h(z) = a(z) + \beta & (z \in]c, d[).
\end{array}
$$

References

[1] Aczél, J. (1966.) *Lectures on Functional Equations and their applications*. Academic Press, New York-London. Mathematics in Science and Engineering, Vol. 19.

[2] Aczél, J. (1983). Diamonds are not the Cauchy extensionist's best friend. *C. R. Math. Rep. Acad. Sci. Canada*, 5, 259–264.

[3] Aczél, J., Erdös, P. (1965). The non-existence of a Hamel-basis and the general solution of Cauchyś functional equation for nonnegative numbers. *Publ. Math. Debrecen*, 12, 259–265. **https://doi.org/10.5486/PMD.1965.12.1-4.25**

[4] Aczél, J., Dhombres, J. (1989.) *Functional equations in several variables*. Cambridge University Press. **https://doi.org/10.1017/CBO9781139086578**

[5] El Bachraoui, M. (2006). Primes in the interval [2n; 3n]. *Int. J. Contemp. Math. Sciences*, 1(13), 617-621. **https://doi.org/10.12988/ijcms.2006.06065**

[6] Cloud, M. J., Kearfott, R. B., Moore, R. E. (2009). *Introduction to Interval Analysis*. Philadelphia: Society for Industrial and Applied Mathematics (SIAM). **https://doi.org/10.1137/1.9780898717716**

[7] Daróczy, Z., Losonczi, L. (1967). Über die Erweiterung der auf einer Punktmenge additiven Funktionenen. *Publ. Math. Debrecen*, 14, 239–245. **https://doi.org/10.5486/PMD.1967.14.1-4.27**

[8] Davis, E. (1987). Constraint propagation with interval labels. *Artificial Intelligence*, 32(3), 281–331. **https://doi.org/10.1016/0004-3702(87)90091-9**

[9] Erdös, P. (1932). Beweis eines satzes von tschebyschef. *Acta Litt. Univ. Sci., Szeged, Sect. Math.*, 5, 194–198.
[10] Euclid of Alexandria (First edition - 2007, Revised and corrected - 2008) *Elements*, Richard Fitzpatrick's bilingual edition freely downloadable PDF, typeset in a two-column format with the original Greek beside a modern English translation; also available in print as ISBN 978-0-615-17984-1
[11] Glavosits, T. *Short remark to the Rimán's Theorem*. (In preparation)
[12] Glavosits, T., Karácsony, Zs. Existence and uniqueness theorems for functional equations, (It will appear in *Commun. Math.* soon.)
[13] Glavosits, T., Karácsony, Zs. (2021a). Sums of intervals in ordered groups and fields. *Acta Univ. Sap., Math.*, 13, 182–191. **https://doi.org/10.2478/ausm-2021-0010**
[14] Glavosits, T., Karácsony, Zs. (2021b). Sums of intervals in ordered semigroups. *An. Şt. Univ. Ovidius Constanţa, Ser. Mat.*, 29, 187–198. **https://doi.org/10.2478/auom-2021-0025**
[15] Graham R. L., Nesetřil J., Butler S. (Eds.) (2013). *The Mathematics of Paul Erdös II.* Springer. **https://doi.org/10.1007/978-1-4614-7254-4**
[16] Hanson, R. J.: *Interval arithmetic as a closed arithmetic system on a computer*. Technical Memorandum 197, Jet Propulsion Laboratory, Section 314, California Institute of Technology, Pasadena, CA.
[17] Jaulin, L., Kieffer, M., Didrit, O., Walter, É. (2001). *Applied Interval Analysis*. Springer, London. **https://doi.org/10.1007/978-1-4471-0249-6_2**
[18] Kahan, W. (1968). *A more complete interval arithmetic*. Lecture notes for a summer course. University of Toronto, Canada.
[19] Kearfott, R. B. (1996). *Rigorous Global Search: Continuous Problems*. Kluwer, Dordrecht. **https://doi.org/10.1007/978-94-017-1247-7_1**
[20] Kuczma, M. (1985). Second edition: Birkhäuser Basel (2009). *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*. First edition: Uniwersytet Slaski w Katowicach. **https://doi.org/10.1007/978-3-7643-8749-5**
[21] Moore, R. E. (1959). *Automatic error analysis in digital computation*. Technical Report LMSD-48421 Lockheed Missiles and Space Co, Palo Alto, CA.
[22] Moore, R. E. (1962). *Interval Arithmetic and Automatic Error Analysis in Digital Computing.* Department Analysis, Stanford University.
[23] Moore, R. E. (1966). *Interval Analysis*. Englewood Cliff, New Jersey, USA: Prentice-Hall., 35.
[24] Radó, F., Baker, J. A. (1987). Pexider's equation and aggregation of allocations. *Aequationes Math.*, 32, 227–239. **https://doi.org/10.1007/BF02311311**
[25] Rimán, J. (1976). Symposium en Quasigroupes et Équations Fonctionnelles (Belgrade-Novi Sad, 1974) On an extension of pexiders's equation. *Zbornik Radova Mat. Inst. Beograd N. S.*, 1(9), 65–72.
[26] Székelyhidi, L. (1972). The general representation of an additive function on an open point set. *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* 21.