ROBUST INTERVAL ESTIMATORS OF WEIBULL PARAMETERS WITH PROBABILITY INTEGRAL TRANSFORMATION

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Abstract
This article gives a construction of a weighted mean which has good robust properties (qualitative robustness, bounded influence function, high breakdown points). The construction is based on M-functionals with smooth defining functions which are used to control weighting. This method can be applied some other statistical problems but it is used for estimation of the parameters of Weibull-family.

Keywords: *robust estimators, Weibull distribution, probability integral transformation*

1. Introduction

Under certain regularity conditions on the underlying probability distribution, it has been shown that maximum likelihood and other best asymptotically normal sequences of estimators are asymptotically *„optimal"* in that they are consistent, asymptotically unbiased and asymptotically efficient. Any of the best asymptotically normal estimators therefore is a satisfactory solution to the estimation problem provided that

- these criteria of optimality are acceptable,
- sufficiently large samples are available,
- the regularity conditions are met.

Typical regularity conditions for the case of a single unknown parameter are given by Cramer (David and Johnson, 1948) or for the robust case it is discussed by Huber (Hampel et al., 1986).

There are many probability distributions for which the conditions are not satisfied. Included among these are several common distributions, notably the uniform, two-parameter exponential, three-parameter Gamma and Weibull distributions. The latter is of particular concern in the present study. It is the location (or shift) parameter which is the source of the difficulty. Furthermore we know that in many cases the classical optimal estimators are not *"good"* if the set of data points contains *öutliers"* (bad points). Accordingly, the robust methods have been created to modify the classical schemes so that the outliers have much less influence on the final estimates. One of the most satisfying robust procedures is that given by a modification of the principle of maximum likelihood. It should be mentioned that a book (Huber, 1981) by Huber provides an excellent summary the mathematical aspects of robustness.

One of the simplest statistical problems is the location-scale problem on the real line. Given a data set

$$\{x_1, x_2, \dots, x_n\},$$

we are required to specify to numbers T and $S,$ together with upper and lower bounds, which describe the location and the scale, respectively, of the data. In spite of its apparent simplicity, the problem has as yet no satisfactory solution. Most approaches including robust ones are based on a central model F_0

which is assumed to be true or to contain the truth within some small metric ball. Data rarely come accompanied by a central model and when analyzing large numbers of data sets in an automatic manner, such an approach is unwarranted. The estimation of the parameters is a well discussed problem. Mostly, the location parameter μ and the scale parameter σ are estimated by the maximum likelihood (ML) estimators $\hat{\mu}$ and $\hat{\sigma}$, respectively. However, there exists other estimators, which are rather efficient or even unbiased, as linear estimators of order statistics. There are many good results and monographs available in this area; only a few subjectively chosen papers are (Hampel et al., 1986; Huber, 1981; Rieder, 1994).

Our statistical approach is based on a central model G_0 which is assumed to be true or to contain the truth within some small metric ball of distribution functions. Two distribution functions F and G are said to be of the same type if

$$F(x) = G(\sigma x + \mu)$$

where $\sigma > 0$. We shall refer to μ as a location (centering) parameter, to σ as a scale parameter. This is an equivalence relation on the set **F** of probability distribution functions. This relation classifies **F**.

In this paper we discuss some new estimators for the parameters of Weibull distribution. In section 2 we summarize a general robust statistical method for location-scale problem. This the so called probability integral transformation (PIT) method (David and Johnson, 1948). In section 3 we propose an algorithm for estimating the location and the scale simultaneously. In section 4 we describe these robust estimators for the parameters of Weibull distribution. The PIT method is used for the Weibull parameters. In section 5 the method is applied for a special mechanical problem with real data.

2. Probability integral transformation estimators

Our location and scale problem is the following:

Let us assume that $\xi = \sigma\eta + \mu$, where the distribution of the random variable η is $G_0(x)$. Given the sample $\xi_1, \xi_2, \dots, \xi_n$ and the type of distribution G_0, the distribution of the random variable ξ_i is

$$G_0\left(\frac{x-\mu}{\sigma}\right)$$

estimate the location ($\mu \in \boldsymbol{R}$) and scale ($\sigma > 0$) parameters from the sample.

The system of equations for the parameters μ and σ, using Huber's (Huber, 1981) notations, is

$$\sum_{i=1}^{n} \psi\left(\frac{\xi_i - \mu}{\sigma}\right) = 0,$$

$$\sum_{i=1}^{n} \chi\left(\frac{\xi_i - \mu}{\sigma}\right) = 0,$$

where $\psi(x) = G_0(x) - 0.5$, $\chi(x) = \psi^2(x) - \frac{1}{12}$.

Therefore,

$$\sum_{i=1}^{n} \left(G_0\left(\frac{\xi_i - T_n}{s_n}\right) - \frac{1}{2}\right) = 0, \quad \sum_{i=1}^{n} \left(\left(G_0\left(\frac{\xi_i - T_n}{s_n}\right) - \frac{1}{2}\right)^2 - \frac{1}{12}\right) = 0. \tag{2.1}$$

If the solutions T_n and s_n of this system of equations exist, T_n and s_n are called the probability integral transformation (PIT)-estimators of the location and the scale parameters, respectively.

Assume that G_0 is differentiable, strictly monotone increasing and $G_0(0) = 0.5$, then T_n and s_n are well defined, that is, (2.1) has a unique solution with $s_n > 0$.

The main steps of the proof: We follow (Huber, 1981). The Jacobian of the map

$$(t,s) \longrightarrow \left(\int \psi\left(\frac{x-t}{s}\right)F(dx), \int \chi\left(\frac{x-t}{s}\right)F(dx)\right)$$

is

$$-\frac{1}{s}\begin{pmatrix} \int \psi'(y)dF & \int y\,\psi'(y)dF \\ \int \chi'(y)dF & \int y\,\chi'(y)dF \end{pmatrix}$$

with $y = \frac{x-t}{s}$. F is indifferently either the true or the empirical distribution.

We define a new probability measure $F^\star$ by

$$F^\star(dy) = \frac{\psi'(y)}{E_F(\psi'(y))}F(dx);$$

then The Jacobian can be written as

$$-\frac{1}{s}E_F(\psi'(y))\begin{pmatrix} 1 & E_{F^\star}(y) \\ E_{F^\star}\left(\frac{\chi'}{\psi'}\right) & E_{F^\star}\left(y\frac{\chi'}{\psi'}\right) \end{pmatrix}$$

Its determinant

$$\left[\frac{E_F(\psi'(y))}{s}\right]^2 \mathrm{cov}_{F^\star}\left(y, \frac{\chi'}{\psi'}\right)$$

is strictly positive. The existence of a solution now follows from the observation that for each fixed s, the first component of the map has a unique zero at some $t = t(s)$ that depends continuously on s. We now conclude from the intermediate value theorem for continuous functions that the solution exists uniquely. The joint asymptotic distribution of (T_n, s_n) can be derived from a general result of Boos and Serfling (Boos and Serfling, 1980).

The joint distribution of (T_n, s_n) converges to the normal one

$$\sqrt{n}\big((T_n, s_n) - (\mu, \sigma)\big) \to dN(0, \Sigma),$$

where the covariance matrix Σ is given by $\Sigma = C^{-1}S[C^{-1}]^T$.

The matrix

$$C = \begin{pmatrix} E\left(\frac{\partial}{\partial\mu}\psi\left(\frac{\xi-\mu}{\sigma}\right)\right) & E\left(\frac{\partial}{\partial\sigma}\psi\left(\frac{\xi-\mu}{\sigma}\right)\right) \\ E\left(\frac{\partial}{\partial\mu}\chi\left(\frac{\xi-\mu}{\sigma}\right)\right) & E\left(\frac{\partial}{\partial\sigma}\chi\left(\frac{\xi-\mu}{\sigma}\right)\right) \end{pmatrix},$$

and

$$S = \begin{pmatrix} E(\psi^2(\eta)) & E(\psi(\eta)\chi(\eta)) \\ E(\psi(\eta)\chi(\eta)) & E(\chi^2(\eta)) \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{180} \end{pmatrix}$$

if G_0 is symmetric for zero.

The (PIT)-estimators are B-robust, V-robust, qualitatively robust and their breakdown points (for definitions see (Fegyverneki, 1999; Hampel et al., 1986))

$$\varepsilon^*(T_n) = \frac{\delta}{1+\delta} = 0.5, \quad \text{where} \quad \delta = \min \left\{ -\frac{\psi(-\infty)}{\psi(+\infty)}, -\frac{\psi(+\infty)}{\psi(-\infty)} \right\},$$

and

$$\varepsilon^*(s_n) = \frac{-\chi(0)}{\chi(-\infty) - \chi(0)} = \frac{1}{3},$$

(see (Fegyverneki, 2004)).

3. Numerical algorithm to PIT estimators

We propose an algorithm to estimate the location and the scale simultaneously.

Step 1: Pre estimation of location and scale by median (med) and median absolute deviation (MAD), i.e.,

$$T_n^{(0)} = med\{\xi_i\} \quad \text{and} \quad s_n^{(0)} = MAD\{\xi_i\}$$

Step 2: Estimation of location by

$$T_n^{(m+1)} = T_n^{(m)} + \frac{s_n^{(m)} \sum_{i=1}^n \psi\left(\frac{\xi_i - T_n^{(m)}}{s_n^{(m)}}\right)}{n}.$$

Step 3: Estimation of scale by

$$[s_n^{(m+1)}]^2 = \frac{12}{(n-1)} \sum_{i=1}^n \psi^2\left(\frac{\xi_i - T_n^{(m+1)}}{s_n^{(m)}}\right)[s_n^{(m)}]^2.$$

Step 4: Stop or goto step 2.

This method can be applied for the system of equations (2.1).

Since the function G_0 is differentiable and strictly monotone increasing the convergence of this iterative method follows from by Huber's result (Huber, 1981) [Section 7.8].

4. Estimators of Weibull parameters

We consider now the random variable η having the three-parameter Weibull distribution. The distribution function of η is given by

$$F(x; a, b, c) = P(\eta < x) = \begin{cases} 1 - \exp\left(-\left(\frac{x-a}{b}\right)^c\right), & \text{if } x \geq a \\ 0, & \text{if } x < a, \end{cases}$$

where the parameters a, b and c are the location, the scale and the shape, respectively.

If $a = 0$ and $\xi = \ln \ \eta$, then ξ is an "extreme-value" variable (smallest element) having distribution function

$$F(x;\vartheta,\sigma) = P(\xi < x) = \begin{cases} 1 - \exp \left[-\exp \left(\frac{x-\vartheta}{\sigma} \right) \right], & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

with location parameter $\vartheta = \ln \ b$ and scale parameter $\sigma = \frac{1}{c}$. The distribution is called Gumbel-distribution.

This means that the parameters can be estimated if the shape c or the location a is known. Because we can apply the mentioned robust algorithm immediately or after a *log transformation.*

For three-parameter case we propose the minimization of Cramer-von Mises statistic

$$\omega_n{}^2 = \frac{1}{12n} + \sum_{k=1}^{n} \left(F(\xi_k{}^\star) - \frac{k-0.5}{n} \right)^2, \qquad (4.1)$$

where $\xi_k{}^\star$ is the kth element of the ordered sample.

The statistic (4.1) is in its limiting form not generally distribution free. But in the important special cases where the estimated parameter is a location, scale or exponential parameter, the limiting distribution is independent of the particular value of the parameter obtaining, which makes the statistic usable for Weibull distributions.

Let

$$G_0\left(\frac{x-\vartheta}{\sigma}\right) = 1 - \exp \left[-exp\left(\frac{x-\vartheta}{\sigma}\right)\right]$$

The joint asymptotic distribution of (T_n, s_n) can be derived from a general result of (Boos and Serfling, 1980).

THEOREM 1. *The joint distribution of* (T_n, s_n) *converges to a normal one:*

$$\sqrt{n}((T_n, s_n) - (\mu, \sigma)) \to dN(0, \Sigma)$$

where the covariance matrix Σ *is given by*

$$\Sigma = \sigma^2 \begin{pmatrix} 1.1704 & -0.1918 \\ -0.1918 & 0.8110 \end{pmatrix}.$$

The main steps of the proof: We use the general result of (Boos and Serfling, 1980) on the central limit theorem for (T_n, s_n) for proving this theorem. T_n and s_n are the solutions of

$$\sum_{i=1}^{n} \psi\left(\frac{\xi_i - \mu}{\sigma}\right) = 0, \qquad \sum_{i=1}^{n} \chi\left(\frac{\xi_i - \mu}{\sigma}\right) = 0,$$

where $\psi(x) = G_0(x) - 0.5$ and $\chi(x) = \psi^2(x) - \frac{1}{12}$. The assumptions are true for the Gumbel distribution, since ψ is monotone increasing and bounded and χ is bounded and negative at 0.

Thus it remains to derive the asymptotic covariance matrix $\Sigma = C^{-1}S[C^{-1}]^T$. Here

$$C = \begin{pmatrix} E\left(\frac{\partial}{\partial\mu}\psi\left(\frac{\xi-\mu}{\sigma}\right)\right) & E\left(\frac{\partial}{\partial\sigma}\psi\left(\frac{\xi-\mu}{\sigma}\right)\right) \\ E\left(\frac{\partial}{\partial\mu}\chi\left(\frac{\xi-\mu}{\sigma}\right)\right) & E\left(\frac{\partial}{\partial\sigma}\chi\left(\frac{\xi-\mu}{\sigma}\right)\right) \end{pmatrix}$$

and with random variable η, it has probability distribution G_0, where $\mu = 0$ and $\sigma = 1$,

$$S = \begin{pmatrix} E(\psi^2(\eta)) & E(\psi(\eta)\chi(\eta)) \\ E(\psi(\eta)\chi(\eta)) & E(\chi^2(\eta)) \end{pmatrix}.$$

These terms can be evaluated by partial integration. The approximations made by program package Maple 16.

$$C = -\frac{1}{\sigma}\begin{pmatrix} \frac{1}{4} & -.067590711365369542506 \\ \frac{1}{36} & .082593278316773246828 \end{pmatrix}$$

and

$$S = \begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{180} \end{pmatrix}.$$

By comparing the asymptotic variances of the PIT estimators T_n and s_n with the ML estimators $\hat{\mu}$ and $\hat{\sigma}$, respectively, we derive the asymptotic relative efficiencies (ARE). It is known that the ML estimators are asymptotically efficient, having also an asymptotic normal distribution, with covariance matrix denoted by $\hat{\Sigma}$. We give the asymptotic relative eficiencies for MD and ML estimators (see Dietrich and Hüssler (Dietrich and Hüsler, 1996)), too.

We get

$$ARE(T_n, \hat{\mu}) \approx 0.9473 \qquad ARE(\mu^*, \hat{\mu}) \approx 0.9395$$

and

$$ARE(s_n, \hat{\sigma}) \approx 0.7496 \qquad ARE(\sigma^*, \hat{\sigma}) \approx 0.7644.$$

The joint asymptotic relative efficiency

$$ARE(PITE, MLE) \approx 0.6663 \qquad ARE(MDE, MLE) \approx 0.6698.$$

For the robustness, we easily derive the breakdown point $\varepsilon^*(T_n)$ of the location estimator. We find that

$$\varepsilon^*(T_n) = 0.5 = \varepsilon^*(\mu^*).$$

We see that these properties are similar.

Because the function χ is not symmetric we cannot use directly the result of Huber (Rieder, 1994) on the breakdown point $\varepsilon^*(s_n)$. However, using his approach with the gross error model we find in the same way that

$$\varepsilon^*(s_n) = \frac{-\chi(0)}{\chi(-\infty) - \chi(0)} \approx 0.2833.$$

The breakdown point $\varepsilon^*(\sigma^*) \approx 0.2026$.

For the robustness, we easily derive the breakdown point $\varepsilon^*(T_n)$ of the location estimator. We find that

$$\varepsilon^*(T_n) = 0.5.$$

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$$\varepsilon^*(s_n) = \frac{-\chi(0)}{\chi(-\infty) - \chi(0)} \approx 0.2833.$$

THEOREM 2. *If we choose the distribution G_0 with $\mu = \ln \ \ln\ (2)$ and $\sigma = 1$ for the functions ψ and χ then the breakdown point $\varepsilon^*(s_n) = \frac{1}{3}$*

This is the maximum for PIT estimators.

The asymptotic results show that the PIT estimators are less efficient than the ML ones. But the efficiency for finite samples might be quite different. Therefore, we simulated finite samples of size n from the Gumbel distribution with $\mu = 0$ and $\sigma = 1$. From these samples the different estimators were calculated. From 500 and 1000 simulations the mean and the standard deviation were derived for each estimator, which were used for the comparison of their finite sample behaviour. We calculated the correlations of the paired estimators and the mean square error (MSE), too. The following tables show that in pairwise comparison the behaviour of PIT amd ML estimators are rather similar in finite. This is also indicated by their correlations.

Table 4.1 *The number of simulations is 500.*

n	mean(T_n)	st.dev.(T_n)	MSE(T_n)	mean($\hat{\mu}$)	st.dev($\hat{\mu}$)	MSE($\hat{\mu}$)	$r(T_n, \hat{\mu})$
10	-0.0478	0.3331	0.1130	-0.0439	0.3198	0.1040	0.9715
20	-0.0264	0.2434	0.0598	-0.0254	0.2386	0.0575	0.9718
30	0.0005	0.2008	0.0402	0.0016	0.1931	0.0372	0.9736
40	0.0040	0.1729	0.0298	0.0027	0.1665	0.0276	0.9729
50	-0.0123	0.1523	0.0233	-0.0108	0.1474	0.0218	0.9702
100	-0.0001	0.1121	0.0125	0.0008	0.1102	0.0121	0.9723
1000	-0.0014	0.0359	0.0012	-0.0009	0.0348	0.0012	0.9742

n	mean(s_n)	st.dev.(s_n)	MSE(s_n)	mean($\hat{\sigma}$)	st.dev($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$r(s_n, \hat{\sigma})$
10	0.9394	0.2836	0.0839	0.9120	0.2494	0.0698	0.9060
20	0.9744	0.1942	0.0383	0.9612	0.1666	0.0292	0.8893
30	0.9705	0.1602	0.0264	0.9613	0.1405	0.0211	0.8724
40	0.9808	0.1389	0.0196	0.9719	0.1207	0.0153	0.8719
50	0.9885	0.1195	0.0144	0.9838	0.1052	0.0113	0.8464
100	0.9912	0.0897	0.0081	0.9868	0.0763	0.0059	0.8714
1000	0.9987	0.0297	0.0008	0.9981	0.0254	0.0006	0.8764

Table 4.2 *The number of simulations is 1000.*

n	mean(T_n)	st.dev.(T_n)	MSE(T_n)	mean($\hat{\mu}$)	st.dev($\hat{\mu}$)	MSE($\hat{\mu}$)	$r(T_n, \hat{\mu})$
10	-0.0374	0.3388	0.1161	-0.0318	0.3273	0.1080	0.9695
20	-0.0289	0.2456	0.0611	-0.0262	0.2369	0.0567	0.9720
30	-0.0228	0.1933	0.0378	-0.0210	0.1892	0.0362	0.9687
40	0.0007	0.1672	0.0279	0.0012	0.1627	0.0264	0.9712
50	-0.0117	0.1497	0.0225	-0.0114	0.1438	0.0207	0.9718
100	-0.0073	0.1109	0.0123	-0.0047	0.1085	0.0117	0.9712
1000	0.0003	0.0343	0.0011	0.0003	0.0333	0.0011	0.9737

n	mean(s_n)	st.dev.(s_n)	MSE(s_n)	mean($\hat{\sigma}$)	st.dev($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$r(s_n, \hat{\sigma})$
10	0.9506	0.2749	0.0779	0.9279	0.2388	0.0621	0.8857
20	0.9756	0.1936	0.0380	0.9623	0.1650	0.0286	0.8623
30	0.9899	0.1742	0.0304	0.9831	0.1470	0.0218	0.8898
40	0.9895	0.1437	0.0207	0.9834	0.1244	0.0157	0.8785
50	0.9933	0.1277	0.0163	0.9878	0.1107	0.0123	0.8803
100	0.9943	0.0921	0.0085	0.9955	0.0808	0.0065	0.8658
1000	0.9994	0.0289	0.0008	0.9994	0.0253	0.0006	0.8736

5. Application and numerical results

We examined three samples *(h37cn, k37cn, he420cn).* The tables 5.1 and 5.2 contain the numerical results. Notations: maximal correlation coefficient method (MCCD), robust location-scale method with Cramer-von Mises statistic (RCM), Kolmogorov statistic (KS), $\chi^2 -$ statistic (χ^2), ${\omega_n}^2$ statistic (${\omega_n}^2$) (Shapiro and Brain, 1984; Smith and Hoeppner, 1990).

The samples contain the results of investigations of fatigue crack growth rate. At investigations we determine the crack length (α) and the number of cycles (N) for the crack of specimen at cyclic loading. After we calculate the values of fatigue crack growth rate $\left(\frac{d\alpha}{dN}\right)$ and stress intensity factor range ($\triangle K$). We approximate the given sequence of points by Paris-Erdogan law (Paris and Erdogan, 1963) which is often used in practice:

$$\frac{d\alpha}{dN} = C \triangle K^m,$$

where C and m are constants. Furthermore, we calculate the values of the fatigue fracture toughness ($\triangle K_{fc}$) to the instable crack growth by the measure of critical crack length. The samples are given from the values of m and $\triangle K_{fc}$ according to materials. The papers (Lukács and Lovas, 1990; Lukács, 1992) contain these results.

Table 5.1 *Numerical results for MCCD*

$Sample$	n	a	b	c	KS	χ^2	${\omega_n}^2$
h37cn	36	3.09	0.69	1.26	0.66	9.48	0.08
k37cn	32	2.75	0.79	2.53	0.49	1.84	0.04
he420cn	32	2.17	0.47	2.57	0.73	7.89	0.09

Table 5.2 *Numerical results for RCM*

$Sample$	n	a	b	c	KS	χ^2	${\omega_n}^2$
h37cn	36	3.09	0.69	1.12	0.56	2.21	0.04
k37cn	32	2.86	0.65	2.03	0.37	2.72	0.02
he420cn	32	2.21	0.40	2.62	0.55	2.36	0.04

Table 5.3 *Critical values for significance level 0.05*

$Sample$	KS	χ^2	${\omega_n}^2$
h37cn	0.88	7.81	0.46
k37cn	0.88	5.99	0.46
he420cn	0.88	5.99	0.46

6. Summary

We defined robust estimators for the parameters of the extreme value distribution G_0 based on the probability integral method. The proposed estimators have bounded influence functions and high breakdown points. These estimators are robust and consistent, but asymptotically less efficient than the maximum likelihood estimators which are not robust. A simulation study for finite sample size shows that under G_0 the efficiency of these robust estimators is rather similar to the maximum likelihood ones.

By the covariance matrix Σ is given by

$$\Sigma = \sigma^2 \begin{pmatrix} 1.1704 & -0.1918 \\ -0.1918 & 0.8110 \end{pmatrix}$$

we can determine the asymptotic confidence interval with the Gauss-distribution.

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