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DOMINATING SETS IN RANDOM GRAPHS

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Abstract
This work presents some random graphs with their dominating sets found by us. After giving an overview of the literature related to the topic and the route of scientific development leading to the definition of the random graphs, the logical process leading to the dominating sets of random graphs has been presented.

Keywords: *graph, random graph, dominating set*

1. Introduction

Dominating set in graphs and networks have been a central topic in graph theory with numerous applications in computer science and engineering. The random graphs are started by Paul Erdős and Alfred Rényi and several works have been created to the nowdays. Random graphs and dominating sets have been still intensively researched topics. Alon's publication (Alon, 1990) has examined the issue theoretically from the point of view of hypergraphs giving a perspective for further research. Balogh and Zhukovski's publication (Balogh and Zhukovski, 2022) has examined the sizes of large subgraphs of the binomial random graph, which is closely related to our article, even if them methods and techniques are different from those of the present article. The famous paper (Barabasi and Albert, 1999) has been created by Barabasi Albert László and Albert Réka. The title of the paper is „Emergence of scaling in random networks". This paper has begun to examine the scale-free random graphs, which modelled well the development of the Internet. Although the article (Barabasi and Albert, 1999) has not been written with mathematical precision, and even some concepts are not properly defined by the authors, nevertheless, looking back, this article became the description of Internet modelling and accordingly, one of the most cited articles. The works by Béla Bollobás on random graphs are excellent summaries of results related to random graphs (Bollobás, 1998; Bollobás, 2001). Bonato and Wang's paper (Bonato and Wang, 2008) has examined the domination parameters in random graphs. Caro, West and Yuster's (Caro et al., 2000) article has contained important new results on the subject. In the publication (Das and Bharghavan, 1997) Das and Bharghavan have examined the routing in ad-hoc networks using minimum connected dominating sets, which are important for certain applications. Deijfen and Lindholm have published a useful article for growing in article (Deijfen and Lindholm, 2009). Duchet and Meyniel's article (Duchet and Meyniel, 1982) has important results which for the presented article. Duckworth and Mans have examined the connected dominating sets of regular graphs in (Duckworth and Mans, 2009). Durrett in (Durrett, 2007) describes the dynamics of random graphs. Important results can be found in the article (Durrett, 2007) which proved to be useful for the present article.

Paul Erdős and Alfred Rényi have introduced the concept of random graphs and began to investigate them in (Erdős and Rényi, 1959). Already in their first scientific work, they have achieved serious results regarding random graphs. Their work has now become a classic. Today, the model developed by Erdős and Rényi is also called the Erdős-Rényi model after the authors. The results reported in this article have been described in this model.

Erdős and Rényi further considered the work in (Erdős and Rényi, 1959) and the results have been published in the article. Feller's work (Feller, 1957) provides the basis for studying the topic.

Flaxman, Frieze and Vera in (Flaxman et al., 2007) have presented the scale-free random graph process.

Gilbert has given a good asymptotic approximation to the probability of certain events in (Gilbert, 1959). Glebov, Liebenau and Szabó in article (Glebov et al., 2015) have examined the domination number of the random graph. Grenander's book (Grenander, 2008) is a comprehensive work in this topic. Guha and Khuller in article (Guha and Khuller, 1995) have given an approximation algorithms for connected dominating sets. Haynes, Hedetniemi and Slater's works (Haynes et al., 1998a) and (Haynes et al., 1998b) have provided a deeper insight into the properties of dominating sets. Li, Wu and Yang in article (Li et al., 2018) have made a dominating set of a graph connected. Liu, Wang and Guo in article (Liu et al., 2010) have given a connected dominating set construction algorithm for wireless sensor networks. In publication (Lovász, 2012), Lovász has discussed large networks and graph limits. Odor and Thiran have given a sequential metric dimension for random graphs in publication (Odor and Thiran, 2021). Ore has given a good summary of the graph theoretical aspects in work (Ore, 1962). In the article (Wieland and Godbole, 2001), Wieland and Godbole have shown the domination number of a random graph. Wu and Li have shown a calculating connected dominating set for efficient routing in ad hoc wireless networks in publication (Wu and Li, 1999).

Of course there are thousands of research papers on the theoretical side and important applications on the practical side. The goal of this paper is to give a comprehensive overview the random graphs with dominating sets.

First we give an exact definition for a dominating set: given a graph G = (V, E) with vertex set V and edge set E, a dominating set is a subset D ⊆ V such that every vertex in V \ D is adjacent with at least one vertex in D.

Examples.

We give some examples for a dominating set.

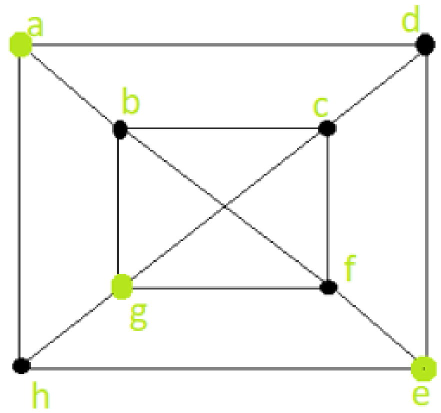

Figure 1.

This graph has 8 vertices and 14 edges. Every black vertex is adjacent to at least one green vertex, which is defined so that the black vertex is dominated by the green vertex. So, the Dominant Set of this graph is S={a,g,e}

We give an another example:

In Figure 2 we show two graphs with different dominating sets (the elements of the dominating set are marked in red). In the first case, the dominant set consists of 1 point, while in the second case, it consists of 8 points. It can therefore be seen that the dominating sets can be very different even within the same graph.

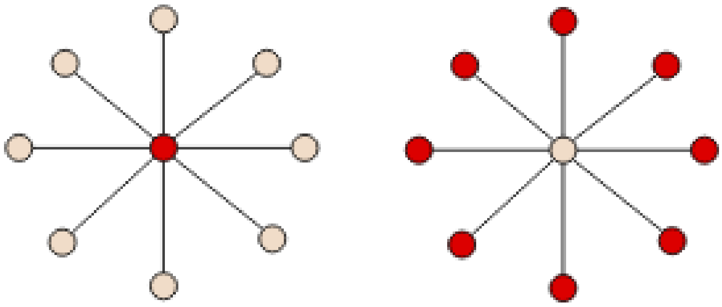

Figure 2.

We give another example:

In Figure 3. there are three given dominating set. It is important to note, that the elements of the dominating set are marked with red color. Thus the definition is So satisfied, that a graph G = (V, E) with vertex set V and edge set E has a dominating set, which is a subset D ⊆ V such, that every vertex in V \ D is adjacent with at least one vertex in D, so here D={the vertex is marked with red color}.

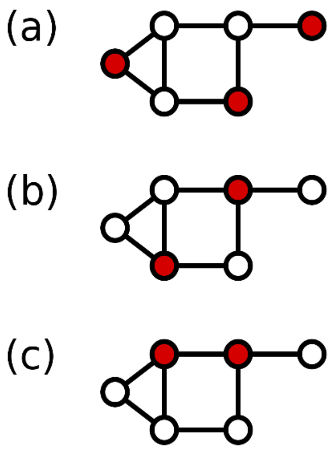

Figure 3.

So the example above (see Figure 3.) the red points are created to be the dominating sets (so we can find more dominating sets in an arbitrary graphs).

In Figure 4. we consider the Petersen graph as well as the dominating set of the Petersen graph (the element of the dominatingΔΔ set we mark with red color).

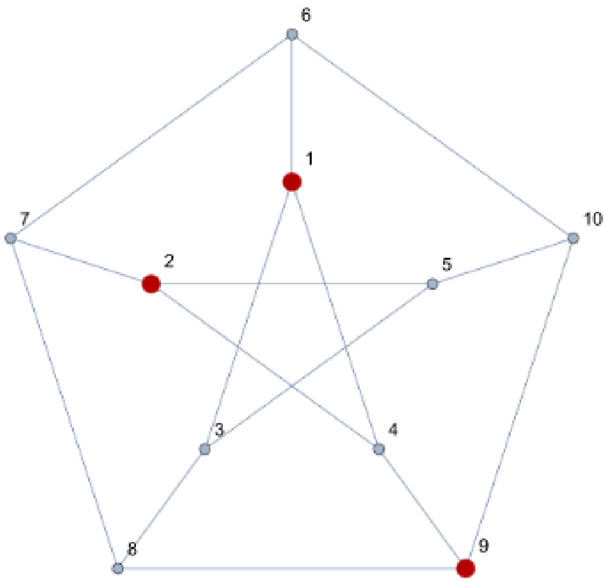

Figure 4.

Remark. The Petersen graph is an undirected graph with 10 vertices and 15 edges. It is a small graph, that serves as a useful example and counterexample for many problems in the graph theory. The Petersen graph has been named after Julius Petersen, who constructed it to be the smallest bridgeless cubic graph with no three-edge-coloring in 1898. Although the graph has been generally attributed to Petersen, it had in fact first appeared 12 years earlier, in a paper created by A. B. Kempe in1886. Kempe has observed, that its vertices can represent the ten lines of the Desargues configuration, and its edges represent the pairs of lines, that do not meet at one of the ten points of the configuration.

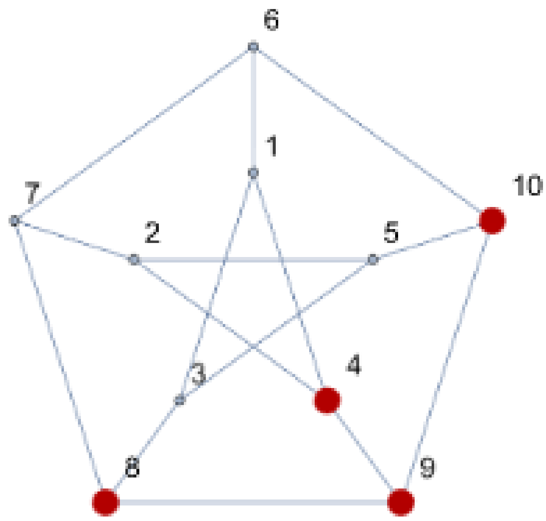

Figure 5.

It can be remarked, that in several works consider the total dominating set (set S of vertices in a graph G(V,E) is called a total dominating set if every vertex v∈V is adjacent to an element of S), just mentioning this fact, which can be seen in Figure 5.

2. The dominating sets of random graphs

Let's consider the random graph $\boldsymbol{G}_{n,p_n}$ on n vertices.

We begin with observations on dominating sets, and finish with connectivity. Let us have an integer function g with $1 \leq g(n) \leq n$. Our aim is to estimate the probability δ_n that a given set X on g(n) vertices dominates the whole $\boldsymbol{G}_{n,p_n}$.

(We have abbreviated the notation, δ_n depends also on g(n).) Let the vertices of the graph be labelled again as $v_1,...,v_n$. First, we give an exact formula for δ_n.

Theorem 1. Let us have an integer function g with $1 \leq g(n) \leq n$. Then we have

$$\delta_n = \left[1-(1-p_n)g(n)\right]^{n-g(n)}.$$

Proof.
Without loss of generality, assume that $X = \{v_1, \dots, v_{g(n)}\}$. Consider any fixed v_j in the range

$$g(n) < j \leq n.$$

Let the exact probability for X to not dominate v_j be denoted by μ_j. Then $\mu_j = P(v_j$ has no neighbor in $\{v_1,...,v_{f(n)}\}) = (1-p_n)^{g(n)}$. Consequently

$$P(\left(\{v_1,\dots, v_{g(n)}\} \textit{ is dominating in } \boldsymbol{G}_{n,p_n}\right)$$

$$= \prod_{j=g(n)+1}^{n} \left[1 - P(X \textit{ does not dominate } v_j)\right]$$

because of the complete independence of the events, constructed from pairwise disjoint sets of edges. The μ_j's have a common value μ. Thus $\delta_n = (1-\mu)^{n-g(n)}$ as stated.

Notation. Let Δ_n denote the probability that there exists a dominating set of cardinality at most g(n) in $\boldsymbol{G}_{n,p_n}$. Furthermore, let

$$\varphi(n) := p_n g(n), s_n := 1/p_n, e_n := [1-1/s_n]^{s_n}, r_n := 1/e_n \textit{ and } F(n) := [n-g(n)]/r_n^{\varphi(n)}$$

The following theorem gives a sufficient condition for $\lim_n \delta_n = \lim_n \Delta_n = 1$.

Theorem 2 If F(n) tends to zero, then δ_n and thus also Δ_n tends to 1.

Proof.
Consider the following relation

$$\delta_n = \left[1-([1-1/s_n]^{s_n})^{\varphi(n)}\right]^{n-g(n)},$$

which can more briefly be written as

$$\delta_n = \left[1-e_n^{\varphi(n)}\right]^{n-g(n)} = \left[1-1/r_n^{\varphi(n)}\right]^{n-g(n)}.$$

Then the denoting $r_n^{\varphi(n)}n$ by t_n,

$$\delta_n = ([1 - 1/t_n]^{t_n})^{F(n)}.$$

By the assumption F(n) → 0 we necessarily have the result, that $r_n^{\varphi(n)}$ tends to infinity; hence

$$[1 - 1/t_n]^{t_n} \to 1/e,$$

and beyond some threshold n_0 we have a form $\delta_n > 1/3^{F(n)}$ for all $n > n_0$. This implies the validity of the theorem.

The theorem below is a very important result in this field.

Theorem 3. For the random graph $\boldsymbol{G}_{n,p_n}$ with n vertices and edge probability p_n, where $(n \cdot p_n - 2 \ln n)$ tends to infinity,
we have the following asymptotic probability of the event that $\boldsymbol{G}_{n,p_n}$ is connected as n → ∞:
The probability then $P\left(\boldsymbol{G}_{n,p_n} \textit{ is connected}\right) \sim 1 - n \cdot (1 - p_n)^{n-1}$.

Proof.
Let us note first that the term $n \cdot (1 - p_n)^{n-1}$ tends to zero as n gets large,
whenever $(n \cdot p_n - 2 \ln n)$ tends to infinity. Indeed, disregarding the multiplier $\frac{1}{1-p_n}$ one may write $(1 - p_n)^n = \left((1 - p_n)^{1/p_n}\right)^{n \cdot p_n} \approx e^{-n \cdot p_n} = n^{-1} \cdot e^{-(n \cdot p_n - \ln n)} = o(n^{-1})$.

Analogously, a similar argument shows that $n \cdot (1 - p_n)^{n/2}$
tends to zero if $(n \cdot p_n - 2 \ln n)$ tends to infinity. Let now $P_n = P\left(\boldsymbol{G}_{n,p_n} \text{ is connected}\right)$.

Instead of P_n we shall estimate $1 - P_n$. Let us introduce the notation $q_n = 1 - p_n$. We claim

$$1 - P_n = \sum_{k=1}^{n-1} P_k \binom{n-1}{k-1} q_n^{k(n-k)}.$$

Indeed, let us fix a vertex, for example the v_0. The whole graph is disconnected if and only if v_0 is contained in a connected subgraph G_0 in such a way that the vertices of G_0 are not joined with any vertex outside. Namely, G_0 is the connected component containing v_0. The order k of G_0 is running between 1 and n − 1, and the set of its vertices can be chosen in $\binom{n-1}{k-1}$different ways.

Any two choices mutually exclude each other, therefore the total probability is equal to the sum of the individual probabilities.

Let E_i^n denote the event that v_i is an isolated vertex, i.e., that v_i is not adjacent to any other vertex in the graph $\boldsymbol{G}_{n,p_n}$.

A lower bound on $1 - P_n$ is the probability $P(E_1^n + E_2^n + \ldots + E_n^n)$ that at least one of the vertices $v_1, v_2, \ldots, v_n$ is isolated. Then

$$1 - P_n \geq P(E_1^n + E_2^n + \ldots + E_n^n) \geq \sum_{i=1}^{n} P(E_i^n) - \sum_{1 \leq j < i \leq n} P\left(E_i^n E_j^n\right) = n q_n^{n-1} - \frac{n(n-1)}{2} q_n^{2n-3}$$

where we applied a simplified version of the inclusion-exclusion principle.

Continuing the proof in a similar way, we obtain the statement of the theorem.

Examples. In both of the following assertions, the condition b > 1 denotes a constant, and the conclusions are derived from Theorem 2.

(i) Let $g(n) = [log_b^{\alpha} n]$ with $\alpha > 1$, and let $p_n = 1/\log_b n$. Then δ_n tends to 1.

(ii) Let $\lim_{n\to\infty} p_n g(n) - \log_b n = \infty$. Then δ_n tends to 1.

3. Summary

In this paper some interesting graphs and their dominating sets have been presented. An investigation had been carried out with random graphs and their dominating sets, and the behavior of dominating set had been presented in random graphs.

Open problem to the future. Are there some p exist tending to zero and some constant b such that

$$\lim_{n\to\infty} P\left(\gamma_c\left(\boldsymbol{G}_{n,p_n}\right) \leq \log_b n\right) > 0?$$

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