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OPTIMIZATION OF KEY JOINTS

Ferenc János Szabó
associate professor, University of Miskolc, Institute of Machine and Product Design
3515 Hungary, Miskolc-Egyetemváros, e-mail: ferenc.szabo@uni-miskolc.hu

Abstract
Key-joints are used in many drives and rotating elements in engineering practice. Nowadays decreasing the dead-weight of the structures is more and more important (in vehicle industry, aircrafts, etc.). This paper shows an optimization process for minimum weight design of key-joints. For the solution of the optimization problem a 3D grapho-analytical optimization process is used, based on the Kuhn–Tucker optimality criterium. During the graphic part of the process, three dimensional diagrams show the behaviour of the objective function and of the design constraints, which makes easy to read the optimum solution from the diagram. The final optimum solution is given in a table numerically too. For better demonstration, the finite element analysis of the three dimensional CAD model of the optimal joint is shown.

Keywords: *Grapho-analytical optimization, key-joints, Kuhn–Tucker optimality criterium, optimization for minimum weight*

1. Introduction

Key-joints are commonly used for power transmission in many types of constructions, rotating elements, drives, clutches, etc. During the design of this kind of elements the engineer selects the dimensions from a table (standards, design specifications, etc.) which can give a feasible solution for the problem, but sometimes it could be useful to find the optimum solution (minimum weight, maximum load carrying capacity, etc.) too. This is why it is possible to find in the litterature several papers optimizing keyways and key connections: (Sahu and Pardeep, 2014) performed the optimization of the keyway design with consideration of effect of stress concentration on different materials; (Zhang et al., 2021) made an optimization study of longitudinal joints. (Váradi and Verghese, 1992) studied the contact state of the key joints very detailly by finite elements. (Pedersen, 2010) made optimization on keyways taking into account the stress concentration effects. The author of this work in several papers (Szabó, 2005, 2017, 2018, 2019) shown the advantages, usefulness and efficiency of the 2D grapho-analytical optimization process in spite of its simplicity. Contact situations are also very important for studies of other machine elements too according to (Sarka, 2022).

In this paper a grapho-analytical 3D optimization method is presented and solved for minimum weight of a key-joint. The optimization problem presented contains three design variables. The mathematical basis of the optimization is the Kuhn–Tucker optimality criterium, saying that at the place of the optimum the contour-line of the objective function is tangent (“touching”) to the feasible area. In two dimensions (2D), this means that a curve will be tangent to an area, but in 3D we will have to search the tangent point between a surface (objective function contour) and the feasible region, which can be a volume. Using this generalization, it is possible to find the optimum solution for many other optimization problems too, having three design variables. The grapho-analytical name of the method means that during the solution, it is possible to investigate the equations and formulae of the design constraints and objective function and find the tangent point by mathematical way, and it is possible to draw the diagrams of these quantities, in order to see better the positions of the constraints and of the objective function which means that it will be easier to understand the structure of the problem, see better the "touching" situation.

These two ways of the solution can help each other and applying both of them, the final solution will be found easier and it will be more accurate. The graphical part of the solution needs to plot 3D graphs of the constraints and the objective function, but using the plot possibilities of several mathematical softwares it will be not too difficult to solve.

2. The structure to be optimized

The 3D model of an assembly containing a key-joint can be seen in *Figure 1*. During the design of this assembly, all the contact state instructions given by (Váradi and Verghese, 1992) also were taken into consideration.

The 3D model of the joint has been analyzed as a numerical example in ANSYS Workbench v.23 R1 academic version of the finite element system (ANSYS Inc, 2011), the total deformation and von Mises equivalent stress contours can be seen in *Figure 2* and *Figure 3*.

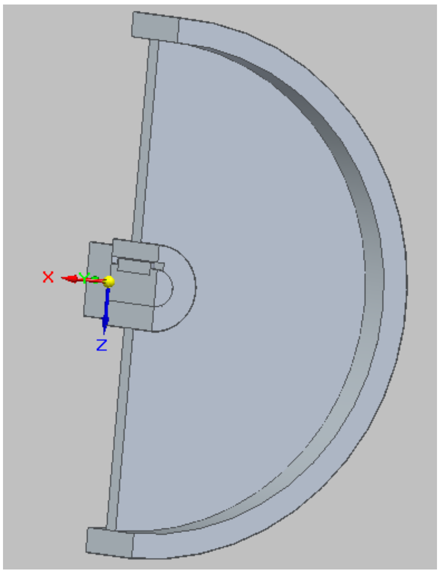

Figure 1. *Shaft and driving element with key-joint*

During the finite element analysis and during the optimization process the load of the structure was a torque of 500 Nm applied on the outer surface of the driver element (rigger), which means $F = 5{,}000$ N force acting on the key surface. The surface pressure (Döbröczöni, 1999) on the key, using the results of the finite element simulation was $p = 82$ MPa. The length of the planar surface of the key (L) was 30 mm, the key width (b) was 6 mm with the height (h) of 4 mm (Nagy, 1991). The value of the shear in the key was $\tau = 42$ MPa. The diameter of the shaft (d) was 20 mm. The material of the key was structural steel, with density of ρ = 7,850 kg/m³. Material of the shaft was C10 steel with R_{eH} = 800 MPa. The material of the driving element also can be steel.

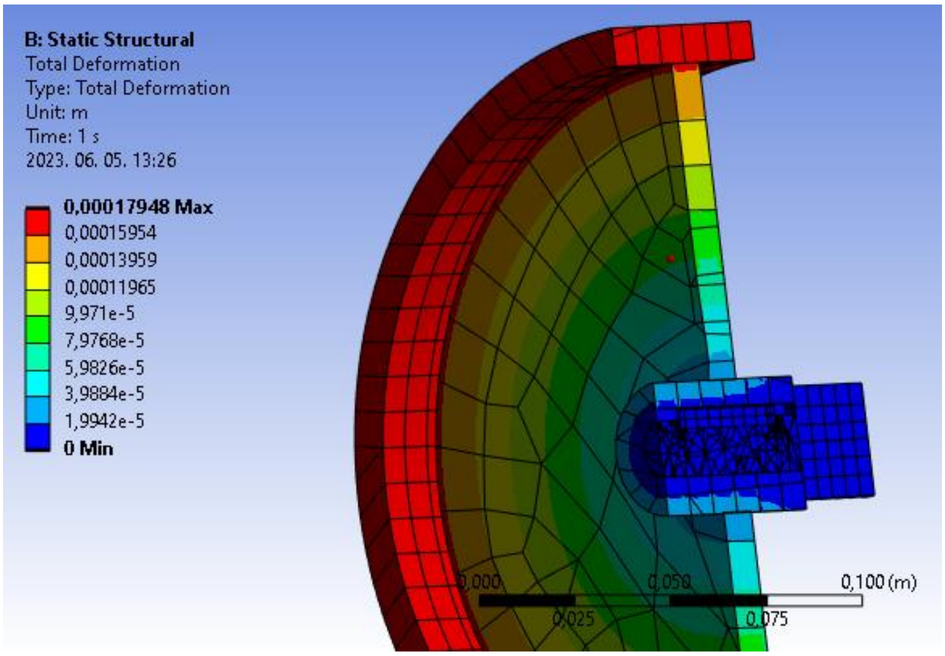

Figure 2. *Contours of deformations*

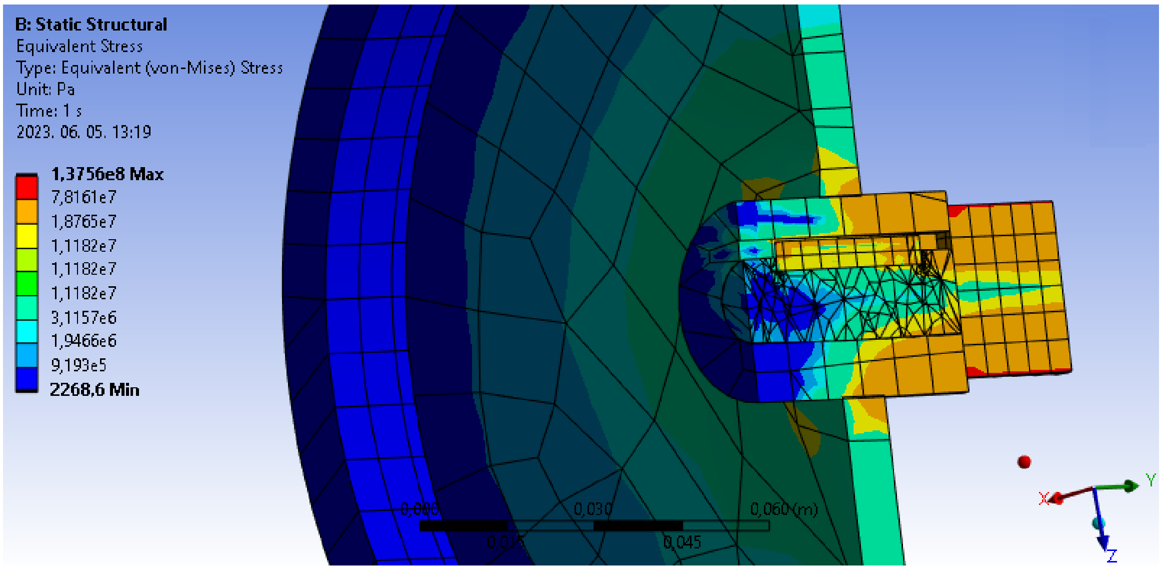

Figure 3. *Contours of stresses*

In the analytical part of the calculations, the following equations were used:

Surface pressure of the key: $p = 2F/(Lh)$ [MPa] (1)

The shear in the key: $\tau = F/(Lb)$ [MPa] (2)

Key cross-sectional area: $A = hb$ [mm^2] (3)

The mass of the key (objective function): $m = hbL\rho$ [kg] (4)

3. The optimization process

For the optimization, a three dimensional, three variables grapho-analytical optimization method has been used, which is based on the Kuhn–Tucker optimality criterion. The main objective of this paper is to demonstrate the application and effectiveness of the three dimensional grapho-analytical optimization and to show a simple numerical example, the key optimization.

The basic idea of the Kuhn–Tucker optimality criterion can be defined easier in two dimensional case: The contour of the objective function (c^*) is tangent to the feasible region in the point of the optimum solution. The contours of the objective function can be generated when we investigate the curves having a constant value of the *f* objective function $f(x_1, x_2) = c$, several constants can be used (c_1, c_2, c_3, c_4, ...). x_1 and x_2 are the design variables of the optimization.

The feasible region is bounded by the curves of the constraints, which can be explicit constraints giving upper or lower limit explicitly to one design variable and implicit constraint can give upper or lower limit to a function $g(x_1,x_2)$ of the design variables (in the *Figure 4* the design variables are denoted by a and b). The functions *f* and *g* can be general nonlinear functions.

Explicit constraints: $l_i \leq x_i \leq u_i$, where *l* and *u* are the lower and upper limits, $i = 1, 2, \ldots, n$ and *n* is the number of the design variables. For the key optimization, the explicit constraints for the key cross section dimensions *b*, *h* are 0 mm as lower limit and 60 mm as higher limit.
Implicit constraints:

$$l_{gj} \leq g(x_1, x_2) \leq u_{gj}, \quad (5)$$

where l_g and u_g are the lower and upper limits of the g function, $j = 1, 2, \ldots, m$, where *m* is the number of the implicit constraints. The implicit constraints for two varables optimization are: Pressure constraint, when the pressure of equation (1) is equal to the maximum permissible pressure value, 100 MPa. Shear constraint, the permissible maximum shear is 75 MPa (Németh, 2023).

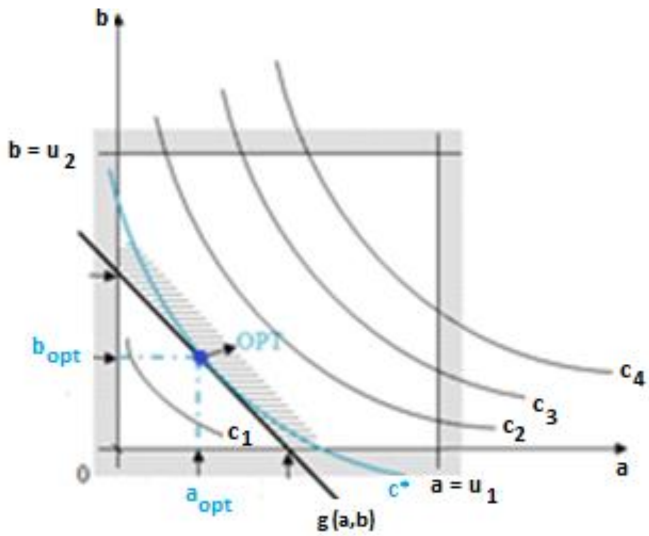

Figure 4. *The feasible region and the objective function contours*

The design variables for the key optimization are the two sizes of the key cross section *(h, b)* and the length of the key *(L)* during three variable optimization, for two variable case the length is constant. The objective function is the mass of the key (4), or in two variable case the objective function can be the area of the key cross-section, too (mass of unit length) during the optimization, the minimum of the objective function will be the goal.

Two variables grapho-analytical optimization

During the two variables optimization, the objective function is the mass per unit length of the key (3). The contours of the objective function can be drawn when constant values of the objective function are regarded (6).

$$A = c = hb \quad ; \quad h = \frac{b}{c} \tag{6}$$

Figure 5 shows the contours of the objective function for several constants *(c_1, c_2, c_3, …, c_n)* in a b, h coordinate system.

Taking into consiederation the explicit and implicit constraints, *Figure 5* shows the opimum solution of the optimization problem. This process shows that it is possible to find the optimum point and optimum solution only by drawing the curves of the constraints and the contours of the objective function. By changing or simplifying the shapes of the equations of the constraints and the objective function, it is possible to simplify this process too, or make the curves easier to draw. This two-phase optimization is the grapho-analytical optimization process, where the graphic and analytical part could help each other to facilitate the optimization. At the end of this process it is very easy to read the optimum solution from the figure. By the analysis of the objective function contour c_n and the implicit constraints, it is possible to solve the touching point coordinates (the optimum value of *b* and *h*) more accurately, even for more difficult situations then presented here.

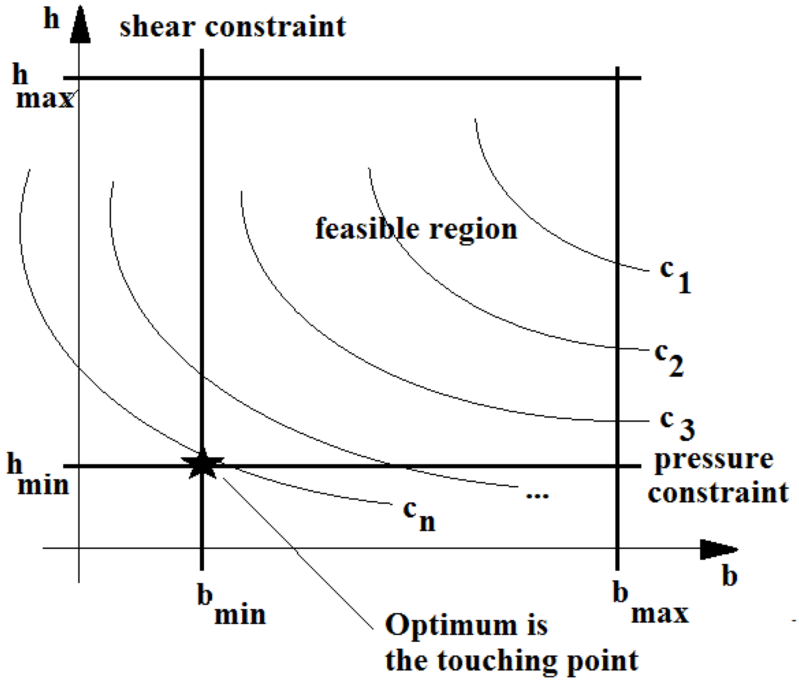

Figure 5. *Two variables grapho-analytical optimization*

This two dimensional optimization process can be generalized into 3D process, taking into consideration the length of the key as a third optimization design variable (in case of 2D process L was supposed as constant).

Three variables grapho- analytical optimization

In case of the three variables optimization, instead of curves it will be necessary to draw surfaces, and it will be necessary to generalize the meaning of "contour of objective function", which will be surface, and also surfaces will be the limits of the constraints, too. Also it could be more general the meaning of "touching" or "tangent" between the feasible region and the contour surface of the objective function, so it would be extremely necessary to apply both the graphical and analytical parts of the solution, helping each side another.

Equation (4) shows the three dimensional form of the objective function and equations (1) and (2) shows the implicit constraints. Explicit constraints remain the same as they were in case of two variables optimization. *Figure 6* shows in three dimensions the objective functions and the constraint surfaces. The three dimensional diagram shows, that the objective function is cutting the constraint surfaces but however it is "tangent" to the feasible region.

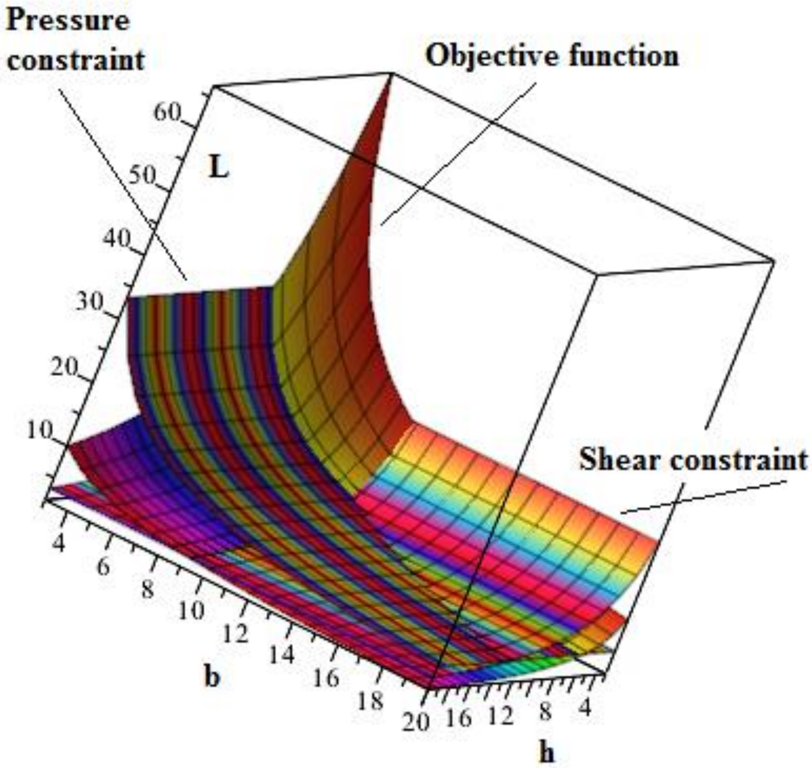

Figure 6. *Three variables grapho-analytical optimization*

In order of better understanding the situation presented in *Figure 6*, *Figure 7* and *Figure 8* highlights some important parts. In *Figure 7* the intersection line gives the optimum height and width of the key in function of the length, this is a generalization of the optimum point of *Figure 5* marked with a star.

Since the reading of points and values sometimes could be less accurate in three dimensional diagrams, it would be useful to see the analytical part of the solution, too.
The objective function:

in 2D: $h = c\frac{1}{b}$, in 3D: $L = \frac{c}{hb}$, where *c* is an arbitrary constant, for the contour of the objective function. A possible way to find the optimum solution could be to iterate with the value of *c*, trying a value and see if it is "tangent" to the feasible region or not. This process is just like we ask if a value of objective function is optimum or not. The value of *c* in case of touching will be the maximum possible achievable value of the objective function. The coordinates of the tangent point are the optimum values of the design variables.

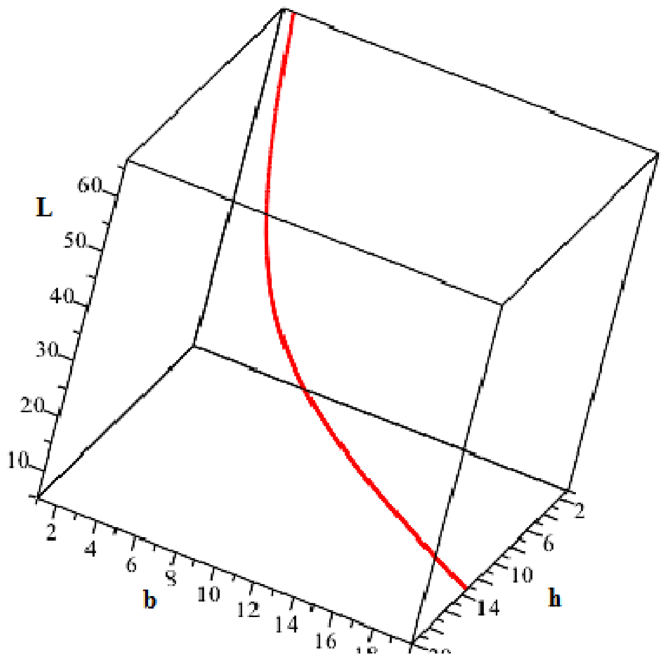

Figure 7. *Intersection line of the constraints surfaces*

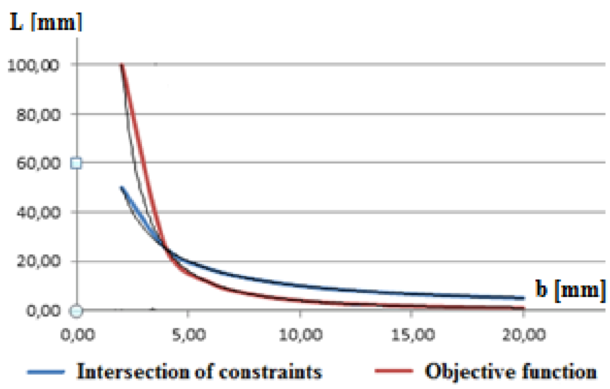

Figure 8. *The objective function and the intersection line*

The explicit constraints are the same in case of 2D or 3D optimization.

The implicit constraints:

in 2D: $$0 \le h \le \frac{F}{2Lp_{adm}} \quad , \quad 0 \le b \le \frac{F}{L\tau_{adm}} \qquad (7)$$

in 3D: $$L \ge \frac{F}{p_{adm}\frac{h}{2}} \quad , \quad L \ge \frac{F}{\tau_{adm}b} \qquad (8)$$

In equations (7) and (8) p_{adm} means the admissible maximum value of the surface pressure on the key and the τ_{adm} means the maximum admissible shear in the key. The numerical results of the optimization are detailed in *Table 1*.

Table 1

Numerical results of the optimization

Method	Key height [mm]	Key width [mm]	Length [mm]	Mass [g]
2D	4	3	30	2.83
2D	5	4	20	3.14
2D	2	2	50	1.57
3D	2	2	50	1.57

4. Conclusions

In this paper the optimization of key-joints for minimum mass is shown, by using the grapho-analytical optimization procedure. In case of two variables a 2D grapho-analytical optimization is realized, in case of three design variables a 3D grapho-analytical optimization is necessary. For 2D problems the design variables are the height and width of the key, in 3D case the third variable will be the length of the key. Objective function for 2D case is the mass per unit length of the key, for three dimensional case the mass of the key is the objective function. The 3D version is a generalization of the 2D process into space, so instead of curves it will be necessary to draw surfaces and instead of intersection point a line of intersection between the surfaces must be considered. The analytical part of the solution could help the graphical process if the equations can be transformed or simplified, which will make it easier to draw the surfaces and to understand the position of the optimum intersection.

According to the theoretical basis of the grapho-analytical optimization (the Kuhn–Tucker optimality criterium), the place of the optimum solution is where the contour of the objective function will be tangent to the feasible region. The feasible region in case of two variables will be a planar area, but in case of three variables it is a volume. The meaning of "tangent" could be also more general in case of 3D optimization then in case of 2D problems.

Comparing the 2D and 3D optimization, the intersection line of the constraint surfaces in case of 3D optimization contains all the possible optimum solution points of 2D problems for different length of the key. One point of this curve is the optimum solution of a 2D problem for a given length. Both in 2D and 3D optimization it is true, that reading the drawings of the constraints and the contours of the objective function, it is very easy to understand the structure of the problem and it is very easy to read the optimum solution from the drawings. Analytical part of the solution will be very useful, in order to increase the accuracy of the optimum results, because reading them from the drawing will summarize the errors of the drawing and the errors of the reading, but in analytical solution the accuracy depends only on the mathematical accuracy of the equations.

The grapho-analytical optimization process, despite of its simplicity, could be very efficient and useful to find important and interesting results for real engineering problems during the design of several machine elements, structural parts or products. On the basis of these optimization results the engineers and designers could considerably improve some important characteristics of the designed element or product. Because of its simplicity, the 2D procedure is very useful in the education of the optimization, helping the students to understand the structure of an optimization problem and find the solution or in graphical or in analytical way.

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