COUNTER-EXAMPLES TO BRECKNER-CONVEXITY

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Abstract
In this paper, we examine convexity type inequalities. Let D be a nonempty convex subset of a linear space, c>0 and α:D-D→R be a given even function. The inequality

$$f\left(\frac{x+y}{2}\right) \le cf(x) + cf(y) + \alpha(x-y) \qquad (x, y \in D)$$

is the focus of our examinations. We will construct an example to show that for c=1, this Jensen type inequality does not imply the convexity of the function. Then, we compare this inequality with Hermite–Hadamard type inequalities.

Keywords: *convexity, Breckner-convexity, counter examples.*

1. Introduction

Denote by $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{R}_+$ the sets of real numbers, positive integers, and nonnegative real numbers, respectively. Let D be a nonempty convex subset of a linear space X and denote by D^* the set $\{x - y: x, y \in D\}$. Let $\alpha: D^* \to \mathbb{R}$ be a nonnegative even error function.

The convexity has many applications and many generalizations. In the first step, we consider the following. We say that a function $f: D \to \mathbb{R}$ is α-Jensen convex, if for all $x, y \in D$,

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2} + \alpha(x-y). \tag{1}$$

Many authors examined this inequality from many contexts. For example (Házy, 2005 and 2007), (Házy et al., 2004, 2005 and 2009), (Makó et al., 2010, 2011, 2012 and 2013), (Ng et al., 1993), (Páles, 2003), (Tabor et al., 2009a, 2009b, 2010a, 2010b). If α is constant zero, we have the notion of classical Jensen-convexity.

In this paper, we will examine the following Jensen type inequality, which is a kind of generalization of the previous notion. Let $c > 0$. We say that a function $f: D \to \mathbb{R}$ is (c, α)-*Jensen convex* if for all $x, y \in D$,

$$f\left(\frac{x+y}{2}\right) \le cf(x) + cf(y) + \alpha(x-y). \tag{2}$$

When $c \neq \frac{1}{2}$ this inequality was examined by Breckner, Breckner and Orban, Házy, Burai and Házy, Burai, Házy and Juhász (Breckner, 1978; Breckner, 2011; Breckner et al., 1978; Házy, 2012; Burai et al., 2011a; Burai et al., 2011b).

The concept of *s*-convexity in the second sense was introduced by (Breckner, 1978). A real valued function $f\colon D \to \mathbb{R}$ is called *Breckner-convex* or *s-convex in the second sense*, if

$$f(tx + (1-t)y)) \leq t^s f(x) + (1-t)^s f(y)$$

for every $x, y \in D$ and $t \in [0,1]$, where $s \in]0,1]$ is a fixed number. If $t = \frac{1}{2}$, then we get the Breckner-Jensen convex function is $(c, 0)$-Jensen convex, with $c = \left(\frac{1}{2}\right)^s$.

In concept of *h*-convexity was introduced by Varošanec (Varošanec, 2007) and was generalized by Házy (Házy, 2011). We say that $f\colon D \to \mathbb{R}$ is an *h*-convex function if, for all $x, y \in D$ and $t \in [0,1]$, we have

$$f(tx + (1-t)y)) \leq h(t)f(x) + h(1-t)f(y).$$

If $t = \frac{1}{2}$, then we get the *h*-Jensen convex function is $(c, 0)$-Jensen convex, with $c = h\left(\frac{1}{2}\right)$.

The Godunova-Levin functions was investigated by (Godunova-Levin, 1985). We say that function $f\colon D \to \mathbb{R}$ (where *D* is a real interval) is a Godunova-Levin function, if *f* is nonnegative and for all $x, y \in D$ and $t \in]0,1[$,

$$f(tx + (1-t)y)) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

If $t = \frac{1}{2}$, then we get the Godunova-Levin-Jensen convex function is $(c, 0)$-Jensen convex, with $c = 2$.

The following theorem is the famous Bernstein-Doetsch theorem (Bernstein et al. 1915).

Theorem 1. *Let I be a nonvoid interval, $f\colon I \to \mathbb{R}$ be locally bounded from above on I and assume that f is Jensen-convex, then f is convex.*

In Section 2, we will prove that if $c \geq 1$, this connection is not valid between (c, α)-Jensen convexity and convexity type inequality.

Now let us recall the theorem of Nikodem, Riedel, and Sahoo from (Nikodem et al., 2007). They proved that from an approximate convexity on an interval I, that is

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon \qquad (x, y \in I),$$

we can get Hermite–Hadamard type inequalities, namely,

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f(tx + (1-t)y)dt + \varepsilon \qquad (x, y \in I),$$

and

$$\int_0^1 f\,(tx+(1-t)y)dt \le \frac{f(x)+f(y)}{2}+\varepsilon \qquad (x,y\in I).$$

But the converse implications are not true. In fact, they constructed some counterexample. In Section 3, we would like to comprise the new generalized Jensen-convexity type inequality ((c,α)-Jensen convexity) and Hermite–Hadamard type inequalities and we will also construct some counterexamples.

2. Counterexamples concerning Bernstein–Doetsch theorem

For the sake of simplicity, assume that $X=\mathbb{R}$ and $D=I$ is a real interval of $\mathbb{R}$ and $\alpha=0$. Then (2) reduces to,

$$f\left(\frac{x+y}{2}\right) \le cf(x)+cf(y) \qquad (x,y\in I).$$

In the following, we will call this inequality c-*Jensen inequality*. With the substitution $x=y$, we have that $0\le(2c-1)f(x)$. This means that if $c>\frac{1}{2}$ then $f(x)\ge 0$ $(x\in I)$ and if $c<\frac{1}{2}$ then $f(x)\le 0$ $(x\in I)$. We will consider the first case. We are looking for functions $\varphi\colon [0,1[\to\mathbb{R}$ such that, for all $t\in[0,1[$ and $x,y\in I$, f satisfies the following convexity type inequality:

$$f(tx+(1-t)y)\le\varphi(t)f(x)+\varphi(1-t)f(y).$$

In the sequel, we will construct an example, which shows, there are no such functions in the case $c\ge 1$.

First, we start some propositions.

Proposition 1. *Assume that a function* $f\colon I\to\mathbb{R}$ *is nonnegative and monotone increasing, then it is also* 1-*Jensen convex.*

Proposition 2. Assume that $f\colon I\to\mathbb{R}$ is 1-Jensen convex, then, for all $d>0$, $f+d$ is also 1-Jensen convex.

Proposition 3. *Let* $\frac{1}{2}\le c\le d$ *Assume that* $f\colon I\to\mathbb{R}$ *is* c-*Jensen convex, then, it is also* d-*Jensen convex.*

Let's consider our first example, namely, for $n\in\mathbb{N}$ and $x\in[0,1[$ let,

$$f_n(x):=\sum_{k=0}^{n}\left([2^{k+1}x]-2[2^kx]\right), \qquad 0\le x<1. \tag{3}$$

Remark 4. It is easy to see that the function $f_n(x)$ is the number of 1's of the binary form of $[2^{n+1}x]$.

The following picture will show the graph of f_n, when $n=5$.

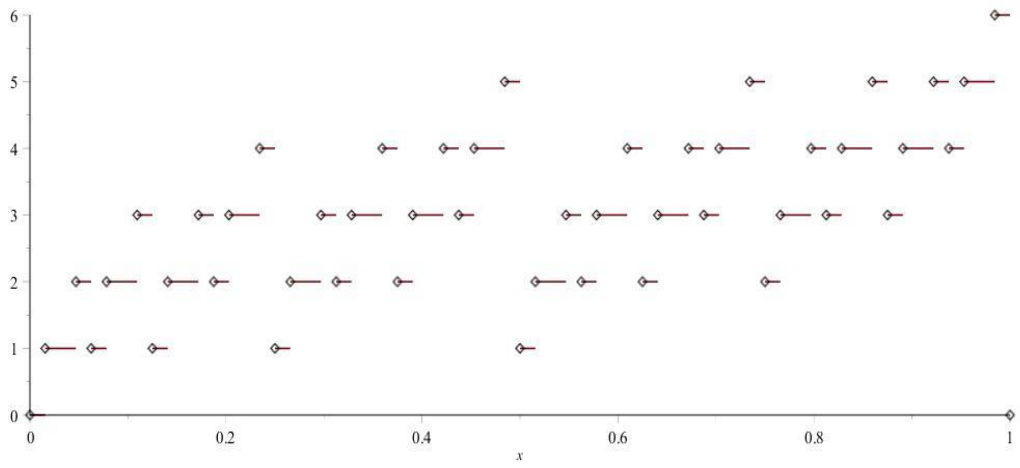

Figure 1. *Graph of function f_5*

Theorem 5. *The function $f_n: [0,1[\to \mathbb{R}$ defined by (3) is 1-Jensen convex, but not convex , i.e., for all $n \in \mathbb{N}$, there exist $\lambda_n \in \mathbb{R}$, with $\lim_{n\to\infty}\lambda_n = \infty$, $t_n \in]0,1[$ and $x_n, y_n \in [0,1[$, such that*

$$f_n(t_n x_n + (1-t_n)y_n) > \lambda_n f_n(x_n) + \lambda_n f_n(y_n).$$

3. Hermite–Hadamard type inequalities and (c,α)-Jensen convexity

In the sequel, we will use the following notion. We say that a function $f: D \to \mathbb{R}$ has got a *radially property*, if for all $x, y \in D$, the function $g_{x,y}: [0,1] \to \mathbb{R}$ defined by

$$g_{x,y}(t) = f(tx + (1-t)y) \qquad t \in [0,1]$$

has got the property. For example, f is *radially bounded*, if for $x, y \in D$, the function $g_{x,y}$ is bounded. Theorem 6 and Theorem 7 show that (c,α)-Jensen convexity implies Hermite–Hadamard type inequalities.

Theorem 6. *Let $c > 0$ and $\alpha: D^* \to \mathbb{R}$ be nonnegative even error function, with for all $u \in D^*$, the map $s \mapsto \alpha(su)$ is Lebesgue integrable on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.*

If $f: D \to \mathbb{R}$ is radially Lebesgue integrable and (c,α)-Jensen convex, then it is also satisfies the following lower Hermite–Hadamard type inequality:

$$f\left(\frac{x+y}{2}\right) \le 2c\int_0^1 f(tx + (1-t)y)dt + \int_0^1 \alpha\big((1-2t)(x-y)\big)dt \qquad (x, y \in D).$$

Theorem 7. *Let $0 < c < 1$ and $\alpha: D^* \to \mathbb{R}$ is an error function, with for all $u \in D^*$, the map $s \mapsto \alpha(su)$ is Lebesgue integrable on $[0,1]$. If $f: D \to \mathbb{R}$ is radially Lebesgue integrable and (c,α)-Jensen convex, then it is also satisfies the following upper Hermite–Hadamard type inequality:*

$$\int_0^1 f(tx + (1-t)y)dt \le \frac{c}{1-c}f(x) + \frac{c}{1-c}f(y) + \int_0^1 \alpha\big(t(x-y)\big)dt \qquad (x, y \in D).$$

In the following theorem, we construct a function, which satisfies a lower Hermite–Hadamard type inequality, but, for all $n \in \mathbb{N}$, it is not n-Jensen convex.

Theorem 8. *Let* $c \geq \frac{3}{2}$. *The following function* $f(x) = x(1-x), \qquad (x \in [0,1])$ *satisfies the lower Hermite–Hadamard type inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{c}{y-x}\int_x^y f(t)dt \qquad (x, y \in [0,1]),$$

but it is not n*-Jensen convex, that is there exists* $x, y \in [0,1]$ *such that*

$$f\left(\frac{x+y}{2}\right) > nf(x) + nf(y).$$

In the following theorem, for all $n \in \mathbb{N}$, we construct a function, which satisfies an upper Hermite–Hadamard type inequality, but, for all it is not n-Jensen convex.

Theorem 9. *For* $n \in \mathbb{N}$, *let*

$$f_n(x) := -ln(|x| + e^{-2n}) + 1, \quad if \qquad |x| \leq 1 - e^{-2n}.$$

Then, for $n \in \mathbb{N}$, f_n *is a continuous function which satisfies the following upper Hermite–Hadamard type inequality,*

$$\frac{1}{y-x}\int_x^y f(t)dt \leq f(x) + f(y) \qquad x < y$$

but it is not $(n, 0)$-Jensen convex, i.e. there exists x, y such that

$$f\left(\frac{x+y}{2}\right) > nf(x) + nf(y)$$

Open problem. *Investigating the Hermite–Hadamard type inequalities,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{c_1}{y-x}\int_x^y f(t)dt \qquad (x < y,\ x, y \in I)$$

and

$$\frac{1}{y-x}\int_x^y f(t)dt \leq c_2 f(x) + c_2 f(y) \qquad (x < y,\ x, y \in I)$$

The case $1 < c_1 < \frac{3}{2}$ and $\frac{1}{2} < c_2 < 1$ are open problems.

We suspect that, counter-examples can be constructed in also these cases.

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