ON PEXIDER ADDITIVE FUNCTIONAL EQUATIONS

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Absztrakt
Legyen Y(+) egy Abel csoport, $D\subseteq Z^2$ ($Z(+,\leq)$ jelöli az egész számok rendezett csoportját). $D_x:=\{u\epsilon Z \mid$ létezik $v\epsilon X$; $(u,v)\epsilon D\}$, $D_y:=\{v\epsilon Z \mid$ létezik $u\epsilon X$; $(u,v)\epsilon D\}$, $D_{x+y}:=\{z\epsilon Z \mid$ létezik $(u,v)\epsilon D$; $z=u+v\}$. A cikk célja megadni azokat a $D\subseteq Z^2$ halmazokat, melyekre az általános megoldása az $f(x+y) = g(x) + h(y)$ ($(x,y \epsilon D)$ függvényegyenletnek $f\colon D_{x+y}\to Y$, $g\colon D_x\to Y$; $h\colon D_y \to Y$ ismeretlen függvényekkel $f(u)=a(u)+C_1+C_2$ ($u\epsilon D_{x+y}$), $g(v)=a(v)+C_1$ ($v\epsilon D_x$); $h(z)=a(z)+C_2$ ($z\epsilon D_y$) alakúak, ahol $a:Z\to Y$ egy additív függvény $C_1, C_2 \epsilon Y$ konstansokkal).

Kulcsszavak: *additív függvények, additív függvényegyenlet, Pexider additív függvényegyenletek, korlátozott Pexdetr additív függvényegyenletek, archimédészien rendezett Abel-csoportok, rendezett sűrű csoportok, függvényegyenletek általános megoldása, diszkrét halmazok*

Abstract
Let Y(+) egy Abelian group, $D\subseteq Z^2$ ($Z(+,\leq)$ denotes the ordered group of the integers). $D_x:=\{u\epsilon Z \mid$ exists $v\epsilon X$; $(u,v)\epsilon D\}$, $D_y:=\{v\epsilon Z \mid$ exists $u\epsilon X$; $(u,v)\epsilon D\}$, $D_{x+y}:=\{z\epsilon Z \mid$ exists $(u,v)\epsilon D$; $z=u+v\}$.The aim of the article is to find sets $D\subseteq Z^2$ that the general solution of the functional equation $f(x+y) = g(x) + h(y)$ ($(x,y \epsilon D)$ with unknown functions $f\colon D_{x+y}\to Y$, $g\colon D_x\to Y$; $h\colon D_y \to Y$ is in the form of $f(u)=a(u)+C_1+C_2$ ($u\epsilon D_{x+y}$), $g(v)=a(v)+C_1$ ($v\epsilon D_x$); $h(z)=a(z)+C_2$ ($z\epsilon D_y$) where $a:Z\to Y$ is an additive function, $C_1, C_2 \epsilon Y$ are constants).

Keywords: *additive functions, additive functional equations, Pexider additive functional equations, restricted Pexider additive fumnctional equations, Archimedean ordered Abelian groups, ordered dense groups, general solution of functional equations, discrete sets*

1. Introduction

The main purpose of this article is to find nonempty sets $D \subseteq \mathbb{Z}^2$ such that the general solution of the functional equation

$$f(x+y) = g(x) + h(y) \qquad (x, y \in D) \tag{PexAdd}$$

with unknown functions $f\colon D_{x+y} \to Y$, $g\colon D_x \to Y$, $h\colon D_y \to Y$, where $Y(+)$ is an Abelian group, the sets D_x, D_y, D_{x+y} are defined by

$$
\begin{aligned}
D_x &:= \{u \in X \mid \exists v \in \mathbb{G}: (u,v) \in D\},\\
D_y &:= \{v \in Y \mid \exists u \in \mathbb{G}: (u,v) \in D\},\\
D_{x+y} &:= \{z \in X \mid \exists (u,v) \in D: z = u + v\},
\end{aligned}
$$

is in the form of

$$
\begin{aligned}
f(u) &= a(u) + C_1 + C_2 & u &\in D_{x+y},\\
g(v) &= a(v) + C_1 & v &\in D_x,\\
h(z) &= a(z) + C_2 & z &\in D_y,
\end{aligned}
\qquad \text{(PexAddMo)}
$$

where $a: \mathbb{Z} \to Y$ is an additive function, $C_1, C_2 \in Y$ are constants.

Definition 1.1. The nonempty set $D \subseteq \mathbb{Z}^2$ is said to be suitable if (preserved the notation above) the general solution of functional equation (PexAdd) is in the form of (PexAddMo).

- If $X(+)$ and $Y(+)$ are structures, $a: X \to Y$ is a function such that $a(x+y) = a(x) + a(y)$ for all $x, y \in X$, the function a is said to be **additive function** (Aczél et al., 1965), (Aczél, 1966), (Aczél et al., 1989) and (Kuczma, 2009).
- If $X(+) = Y(+) = \mathbb{R}(+)$, and function $a: \mathbb{R} \to \mathbb{R}$ is an additive function, then function a is said to be **Cauchy additive function** (Cauchy, 1821), see also (Legendre, 1791; Gauss, 1809).
- It is worth mentioning that A. L. Cauchy was the first who proved that the continuous additive function $a: \mathbb{R} \to \mathbb{R}$ is in the form $a(x) = cx$ for all $x \in \mathbb{R}$ where $c \in \mathbb{R}$ is a constant. It is well-known that if a Cauchy additive function a is not continuous, then the graph of a is dense in $\mathbb{R}^2$ which can be easily show by Hamel bases.
- If $X(+)$, and $Y(+)$ are structures, $D \subseteq X^2$, and the unknown functions $f: D_{x+y} \to Y$, $g: D_x \to Y$, $h: D_y \to Y$ satisfy the functional equation (PexAdd), then the equation (PexAdd) is said to be **restricted Pexider additive functional equation** (Kuczma, 2009; Rimán, 1976; Glavosits, 2023).
- The results concerning to the restricted additive (but not Pexider additive) functional equations, where $D \subseteq \mathbb{R}^{2N}$ is a nonempty connected open set, $Y(+) = \mathbb{R}^n(+)$ can be found in the book (Kuczma, 2009). In the paper (Radó et al., 1987) can be found a Pexider additive functional equation with similar settings and similar general solution to the previous one.
- Worth mentioning the articles (Rimán, 1976) which deal with a Pexider-additive functional equation. This equation is fulfilled on a nonempty connected open set $D \subseteq \mathbb{R}^2$, the codomain of the unknown functions are an Abelian group $E(+)$. In (Glavosits, 2023) the author deals with both restricted additive functional equations, and restricted Pexider additive functional equations with more abstract algebraic settings.
- An $X(\leq)$ **structure is said to be ordered set**, if the relation $\leq$ is reflexive, anti-symmetric, transitive, and linear (i. e. $x \leq y$, or $y \leq x$ is fulfilled for all $x, y \in X$).
- If $X(\leq)$ is an ordered set, $a, b \in X$ such that $a < b$, then the set $]a, b[:= \{x \in X \mid a < x < b\}$ is said to be **open interval** (in X). (There are no any other sets which is said to be open interval.) Similarly $[a, b] := \{x \in X \mid a \leq x \leq b\}$.
- **An ordered set** $X(\leq)$ **is said to be dense** (in itself), if $]a, b[\neq \emptyset$ for all $a, b \in X$ with $a < b$.
- **An ordered group** $\mathbb{G}(+, \leq)$ **is said to be Archimedean ordered**, if for all $x, y \in \mathbb{G}_+ := \{x \in \mathbb{G} \mid x > 0\}$ there exists an $n \in \mathbb{Z}_+$ such that $y < nx := x + \cdots + x$.

2. Some properties of $\mathbb{Z}$

Theorem 2.1. *If $X(+,\leq)$ is an Archimedean ordered Abelian group which is not dense, then the group X is isomorphic to the ordered group $\mathbb{Z}(+)$.*

Since the topology on $\mathbb{Z}$ generated by the open intervals results discrete topology thus we have to break with the usual terminology, according to which we interpret the Pexider additive functional equation on well-chained open sets (Glavosits et al., 2023). For this purpose we introduce a new notation

$$[a,b] := \{a, a+1, \dots, b\}$$

for all a, $b \in \mathbb{Z}$ with $a \leq b$, allowing that $[a,b]$ is a singleton whenever $a = b$.

Proposition 2.1. If a, b, c, $d \in \mathbb{Z}$ with $a \leq b$, and $c \leq d$, then

$$[a,b] + [c,d] = [a+c, b+d].$$

3. Additive functions on $\mathbb{Z}$

If $Y(+)$ is an arbitrary group, $\in Y$ $x \in \mathbb{Z}$, then the element $cx \in Y$ is defined by

$$cx := \begin{cases} c + \cdots + c, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ (-c) + \cdots + (-c) & \text{if } x < 0. \end{cases}$$

Theorem 3.1. *If $Y(+)$ is an Abelian group, $a\colon \mathbb{Z} \to Y$ is an additive function, then there exists an element $c \in Y$ such that $a(x) = cx$ for all $x \in \mathbb{Z}$.*

Example 3.1. Let $a\colon \mathbb{Q} \to Y$ be an additive function and let $Y(+)$ be an Abelian group, then by the $\mathbb{Q}$ homogenicity of additive functions (Kuczma, 2009) there exits a constant $c \in Y$ such that

$$a(x) = cx \qquad (x \in \mathbb{Q})$$

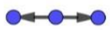
where $c = a(1)$.

Conversely, if $c \in Y$, and the function $a\colon \mathbb{Q} \to Y$ is defined by $a(x) := cx$ for all $x \in \mathbb{Q}$, then the function a is additive.

This Examples shows an interesting analogy to the Theorem 3.1.

Theorem 3.2. *If $Y(+)$ is an Abelian group, $n \in \mathbb{Z}_+$ is a fixed constant, $I := [-n,n]$, the function $f\colon [-2n, 2n] \to Y$ satisfies the functional equation $f(x+y) = f(x) + f(y)$ for all $x, y \in [-n,n]$, then there exists an additive function $a\colon \mathbb{Z} \to Y$ such that $a(x) = f(x)$ for all $(x \in I)$ (i.e. there exists an additive extension of the function f from the set I^2 to the set $\mathbb{Z}^2$).*

In the sequel we shall use the well-known concept of neighbourhood.



The numbers x, $y \in \mathbb{Z}$. is said to be neighbour, if |x-y|=1.

Every integers have two neighbours.

Figure 1

Theorem 3.3. *If $Y(+)$ is an Abelian group, $a, b \in \mathbb{Z}$ such that $a < b$, $I := [a,b] \subseteq \mathbb{Z}$, c_i, $C_i \in Y$ $(i = 1,2)$, and $f: I \to Y$ is a function such that*

$$\begin{aligned} f(x) &= c_1 x + C_1 \qquad (x \in I), \\ f(x) &= c_2 x + C_2 \qquad (x \in I), \end{aligned}$$

then $c_1 = c_2$, and $C_1 = C_2$.

The above Theorem 3.2. and the Theorem 3.3. shows a close analogy to the adequate Theorems in (Glavosits et al., 2023b) concerning Archimedean ordered dense Abelian groups.

4. Some examples

We shall use the following notations in the sequel.

- $B(x_0, n) := [x_0 - n, x_n + n]$ for all $x_0 \in \mathbb{Z}$, and $n \in \mathbb{Z}_+ \cup \{0\}$.
- $B\big((x_0, y_0), n, m\big) := B(x_0, n) \times B(y_0, m)$, and $B\big((x_0, y_0), n\big) := B\big((x_0, y_0), n, n\big)$.
- A subset $R \subseteq \mathbb{Z}^2$ is said to be $m \times n$ type rectangular if there exists $(x_0, y_0) \in \mathbb{Z}^2$ such that $R := [x_0, x_0 + n] \times [y_0, x_0 + m]$. (Imagine the points of the rectangular arranged in rows and columns, similarly to matrices.)

If $D := B\big((x_0, y_0), m, n\big) \subseteq \mathbb{Z}^2$, then $D_x = B(x_0, m)$, $D_y = B(y_0, n)$, and by Proposition 2.1 $D_{x+y} = D_x + D_y = B(x_0 + y_0, m + n)$.

Theorem 4.1. *Let $n \geq 1$, and $D := B\big((x_0, y_0), n\big)$. If the functions $f: D_{x+y} \to Y$, $g: D_x \to Y$, $h: D_y \to Y$ satisfy the functional equation (PexAdd) then it is in the form of (PexAddMo) i.e. there exist constants c, C_1, $C_2 \in Y$ such that*

$$\begin{aligned} f(u) &= cu + C_1 + C_2 \qquad (u \in D_{x+y}), \\ g(v) &= cv + C_1 \qquad (v \in D_x), \\ h(z) &= cz + C_2 \qquad (z \in D_y), \end{aligned}$$

n other words the 3×3 type rectangles are suitable sets for all $n \geq 3$.

Theorem 4.2.

1. *If the set $D \subseteq \mathbb{Z}^2$ is suitable, then the sets*

$$D + (x, y) := \{(u + x, v + y) \mid (u, v) \in D\}$$

   are also suitable for all $x, y \in \mathbb{Z}$.
2. *If the sets D^1, $D^2 \subseteq \mathbb{Z}^2$ are suitable, moreover, the set $D^1_{x+y} \cap D^2_{x+y}$ contain two neighbour elements, $D^1_x \cap D^2_x \neq \emptyset$, and $D^1_y \cap D^2_y \neq \emptyset$, then the set $D^1 \cup D^2$ is also suitable.*

Theorem 4.3. *Let $Y = Y(+)$ be an Abelian group, and the set $D := [a,b] \times [c,d] \subseteq \mathbb{Z}^2$ be an $m \times n$ type rectangular such that $m \geq 1$, and $n \geq 2$. Consider the functional equation (PexAdd) with unknown functions $f: D_{x+y} \to Y$, $g: D_x \to Y$, $h: D_y \to Y$.*

1. *If* $\boldsymbol{m = 1}$*, then the general solution of equation (PexAdd) is in the form*

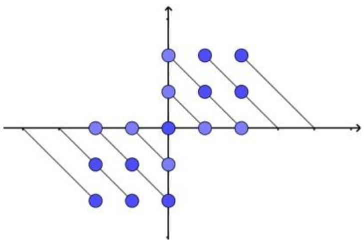
$$\begin{aligned} \boldsymbol{f(x_k + y_1)} &= \boldsymbol{\alpha_k + \beta_k} \\ \boldsymbol{g(x_k)} &= \boldsymbol{\alpha_k} \qquad\qquad (\boldsymbol{k} = \boldsymbol{1, 2, \dots, m}), \\ \boldsymbol{h(y_1)} &= \boldsymbol{\beta_k}; \end{aligned}$$

   where $(\boldsymbol{\alpha_k})$*, and* $(\boldsymbol{\beta_k})$ *are arbitrary sequences, thus the set* $\boldsymbol{D}$ *is not suitable.*
2. *If* $\boldsymbol{m \geq 2}$*, then the general solution of equation (PexAdd) is in the form of (PexAddMo), where* $\boldsymbol{c}$*,* $\boldsymbol{C_1}$*,* $\boldsymbol{C_2} \in \boldsymbol{Y}$ *are constants, i. e.* $\boldsymbol{D}$ *is suitable.*

(The case $m \leq n$ *can be discussed in an analogous way.)*

5. Additional examples

Example 5.1. Let $D^1 := \{0,1,2\}^2$, $D^2 := \{-2,-1,0\}^2$, $D := D^1 \cup D^2$.



$D_x^1 = D_y^1 = \{0,1,2\}$,

$D_{x+y}^1 = \{0,1,2,3,4\}$,

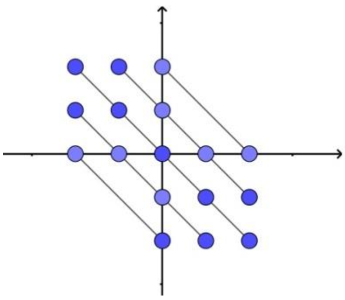
$D_x^2 = \{-2,-1,0\}$,

$D_{x+y}^2 = \{-4,-3,-2,-1,0\}$.

Figure 2

It is easy to see that the set D is not suitable.

Example 5.2. Let $D^1 := \{-2,-1,0\} \times \{0,1,2\}$, $D^2 := \{0,1,2\} \times \{-2,-1,0\}$, and $D := D^1 \cup D^2$.



$D_x^1 = D_y^2 = \{0,1,2\}$,

$D_y^2 = D_x^2 = \{-2,-1,0\}$,

$D_{x+y}^1 = D_{x+y}^2 = \{-2,-1,0,1,2\}$.

Figure 3

It is easy to see that the set D is suitable.

6. Summary. Define the family $\mathcal{D} \subseteq \mathbb{Z}^2$ by

1. The $\mathbf{2} \times \mathbf{2}$ type rectangles are elements of the family $\mathcal{D}$;
2. If $\mathbf{D} \in \mathcal{D}$, and $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^2$, then $\mathbf{D} + (\mathbf{x}, \mathbf{y}) \in \mathcal{D}$.
3. If $\mathbf{D}^1, \mathbf{D}^2 \in \mathcal{D}$, $\mathbf{D}^1_{\mathbf{x+y}} \cap \mathbf{D}^2_{\mathbf{x+y}}$ contains two neighbour elements, $\mathbf{D}^1_{\mathbf{x}} \cap \mathbf{D}^2_{\mathbf{x}} \neq \emptyset$, and $\mathbf{D}^1_{\mathbf{y}} \cap \mathbf{D}^2_{\mathbf{y}} \neq \emptyset$, then the set $\mathbf{D}^1 \cup \mathbf{D}^2$ is also element of the family $\mathcal{D}$.

It is clear that the set family D does not contain all suitable sets.

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