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VARIATIONS ON A THEME

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Abstract
In this paper the general solution of the functional equation f(x+y)=g(xy) is given where the unknown function f, and g satisfy the equation for all x, y from the set A, for different typical choices of the set A. The different choices of the set A require significantly different approaches.

Keywords: *Archimedean ordered fields, functional equations, general solutions of Pexider functional equations, interval aricmetic, Pexider equations, restricted Pexider functional equations, open intervals, ordered fields.*

1. Introduction

The main purpose of this article is to give the general solution of the functional equation

$$f(x+y) = g(xy) \tag{1}$$

for all $x, y \in A$ with unknown functions

$$\begin{aligned} f \quad &: A + A := \{a + b \mid a, b \in A\} \to Y, \\ g \quad &: A \cdot A := \{ab \mid a, b \in A\} \to Y \end{aligned}$$

in the following cases:

1. The set A is $\mathbb{R}_+$ (where $\mathbb{R}_+$ denotes the set of all positive real numbers).
2. The set A is $\mathbb{F}_+$ (where $\mathbb{F}_+$ denotes the set of all positive elements of an Archimedean ordered field $\mathbb{F}(+,\cdot,\leq)$).
3. The set A is a nonempty open interval of the set of all positive elements of an Archimedean ordered field $\mathbb{F}$;
4. The set A is the set of all positive integers;
5. The set A is the set of all positive dyadic rational numbers.

In all of the above cases Y is an arbitrary nonempty (or infinite) set.

There are many articles in which the general solution of function equations similar to the function equation we are investigating is given, and the search for the general solution is set in the form of a problem to be solved.

In (Glavosits et al., 2016) we have shown that if $\mathbb{F}(+,\cdot,\leq)$ is an ordered field, Y is an arbitrary nonempty set, then the functional equation (1) has only constant function solutions on $\mathbb{F}_+$. In (Glavosits et al., 2016) equation (1) is necessary to show that a Pexider type functional equation has only quasi logarithmic function solutions.

In (Glavosits) there is a functional equation that it can be brought back to the so-called generalized Davidson functional equation, by functional equation (1).

In (KöMaL, 2012) (Problem B. 4456.) the investigated functional equation is

$$f\left(\frac{x+y}{2}\right) = f\left(\sqrt{xy}\right) \qquad (x, y \in \mathbb{R}_+) \tag{2}$$

with unknown functions f, $g\colon \mathbb{R}_+ \to \mathbb{R}$. This Problem was proposed by professor Z. Daróczy for secundary school students in the journal KöMaL. The KöMaL, that is, "Középiskolai Matematikai és Fizikai Lapok" (Mathematical and Physical Journal for Secondary Schools) is a Hungarian mathematics and physics journal for high school students. It was founded by Dániel Arany, a high school teacher from Győr, Hungary and has been continually published since 1894.

In the problem book (Small, 2007) (1.13 Problem 2., page 26.) the investigated functional equation is

$$f(x+y) = f(xy) \qquad (x, y \in \mathbb{R}_+) \tag{3}$$

with unknown function $f\colon \mathbb{R}_+ \to \mathbb{R}$. Both of the above two functional equation is similar to functional equation (1).

In this paper a construction is given that if the set Y is an arbitrarily fixed countable set, then there exists a set $A \subseteq \mathbb{R}_+$ that is closed under the addition and multiplication and there exist functions f, $g\colon A \to Y$ that satisfy the functional equation (1) for all $x, y \in A$ and the range of the function f is the set Y.

Now, we give s sort list of some necessary concepts and notations.

6. An ordered group $\mathbb{G}(+,\leq)$ is said to be dense (in itself), if for all x, $y \in \mathbb{G}$ with $x < y$ there exists an element $z \in \mathbb{G}$ such that $x < z < y$.
7. If $\mathbb{G}(+,\leq)$ is an ordered dense group, $]\underline{a},\overline{a}[$, $]\underline{b},\overline{b}[\subseteq \mathbb{G}$, then
$$]\underline{a},\overline{a}[+]\underline{b},\overline{b}[=]\underline{a}+\underline{b},\overline{a}+\overline{b}[;$$
8. If $\mathbb{F}(+,\cdot\leq)$ is an ordered field, $]\underline{a},\overline{a}[$, $]\underline{b},\overline{b}[\subseteq \mathbb{F}_+$, then
$$]\underline{a},\overline{a}[\cdot]\underline{b},\overline{b}[=]\underline{ab},\overline{ab}[$$

(Glavosits et al., 2021a), see also (Glavosits et al., 2021b).

2. Ont he case when A is R+

Theorem 2. 1. *If Y is a nonempty set, the functions f, $g\colon \mathbb{R}_+ \to Y$ satisfy the functional equation (1), then the functions f, and g are constant functions.*

Proof. (Hint.) The function equation can be brought back to the solution of Problem 44456.

Define the functions $F(x) := f(2x)$, $G(x) := g(x^2)$ for all $x \in \mathbb{R}_+$. Thus

$$F\left(\frac{x+y}{2}\right) = G\left(\sqrt{xy}\right) \qquad (x, y \in \mathbb{R}_+). \tag{4}$$

Take $x = y$ thus we have that $F(x) = G(x)$ for all $x \in \mathbb{R}_+$ (which leads us the funtional equation of Ptroblem B. 4456).

$$F\left(\frac{x+y}{2}\right) = F\left(\sqrt{xy}\right) \qquad (x, y \in \mathbb{R}_+). \tag{5}$$

Take the substitution in the equation above

$$x \leftarrow x - \sqrt{x^2 - y^2}, \qquad y \leftarrow x + \sqrt{x^2 - y^2}.$$

Thus we obtain that

$$F(x) = F(y) \qquad (x, y \in \mathbb{R}_+, x > y). \tag{6}$$

which was to be proved. □

Remark 2. 1. The process outlined above cannot be applied in the case where the ordered field is not closed under the square root extraction, thus we need to find a substitution that bypasses the square root extraction.

3. On the case when the set A is the set of all positive elements of an ordered field F(+,·,≤)

Theorem 3. 1. *If* $\mathbb{F}(+,\cdot,\leq)$ *is an ordered field,* Y *is a nonempty set, the functions* $f\colon \mathbb{F}_+ \to Y$*, and* $g\colon \mathbb{F}_+ \to Y$ *satisfy the functional equation (1), then the functions* f*, and* g *are constant functions.*

Proof. It is only a Hint. The proof in more details can be found in the article (Glavosits et al., 2016).

Take the substitution in the equation (1)

$$x \leftarrow \frac{u^2+1}{u}\frac{u}{u^4+u^2+1}, \qquad y \leftarrow \frac{u}{u^2+1}\frac{u^5+u^3}{u^4+u^2+1}.$$

Thus we obtain that

$$\begin{aligned} &f\left(\frac{u^2+1}{u}\frac{u}{u^4+u^2+1} + \frac{u}{u^2+1}\frac{u^5+u^3}{u^4+u^2+1}\right) \\ &= g\left(\frac{u^2+1}{u}\frac{u}{u^4+u^2+1} \cdot \frac{u}{u^2+1}\frac{u^5+u^3}{u^4+u^2+1}\right) \\ &= f\left(\frac{u}{u^4+u^2+1} + \frac{u^5+u^3}{u^4+u^2+1}\right). \end{aligned}$$

Since

$$\begin{aligned} \frac{u^2+1}{u}\frac{u}{u^4+u^2+1} + \frac{u}{u^2+1}\frac{u^5+u^3}{u^4+u^2+1} &= 1, \\ \frac{u}{u^4+u^2+1} + \frac{u^5+u^3}{u^4+u^2+1} &= u, \end{aligned}$$

thus we obtain that $f(u) = f(1)$ for all $u \in \mathbb{F}_+$. □

4. On the case when the set A is an open interval of an Archimedean ordered filed F

Theorem 4. 1. *If $\mathbb{F}(+,\cdot,\leq)$ is an Archimedean ordered field, α, $\beta \in \mathbb{F}$ with $0 \leq \alpha < \beta$, Y is an arbitrarily fixed countable infinite set and the functions $f\colon]2\alpha, 2\beta[\to Y$ and $g]\alpha^2, \beta^2[\to Y$ satisfy equation (1) for all $x, y \in]\alpha, \beta[$, then these functions are constant functions.*

Proof. It is only a Hint. The proof in more details can be found in the article (Erdei, at al., 2023).

Let $x, y \in A + A =]2\alpha, 2\beta[$ such that $x < y$. Then there exists $\varepsilon \in \mathbb{F}_+$ such that $2\alpha < x - \varepsilon$ and $y + \varepsilon < 2\beta$.

Define the sequences (δ_n) and (λ_n) by

$$\delta_n := \frac{y - x}{n}, \quad \lambda_n := \frac{\delta_n + \varepsilon}{\delta_n + 2\varepsilon} \qquad (n \in \mathbb{Z}_+).$$

thus there exists $n_0 \in \mathbb{Z}_+$ such that

$$\frac{a}{x - \varepsilon} < \frac{\varepsilon}{\delta_n + 2\varepsilon} = 1 - \lambda_n.$$

There exists $n_1 \in \mathbb{Z}_+$ such that

$$\lambda_n = 1 - \frac{\varepsilon}{\delta_n + 2\varepsilon} < \frac{\beta}{y + \varepsilon}$$

for all $n > n_1$.

Let $N > \max(n_0, n_1)$ be an arbitrarily fixed integer. Let

$$\lambda := \lambda_N, \qquad \text{and} \qquad \delta := \delta_N,$$

and define the sequence (x_k) by

$$x_k := x + k\delta \qquad (k = 0,1, \dots, N).$$

whence we obtain that

$$\lambda(x_k - \varepsilon), \quad (1 - \lambda)(x_{k+1} + \varepsilon), \quad (1 - \lambda)(x_k - \varepsilon), \quad \lambda(x_{k+1} + \varepsilon) \in]\alpha, \beta[.$$

Simple calculation shows that

$$\begin{aligned} \lambda(x_k - \varepsilon) + (1 - \lambda)(x_{k+1} + \varepsilon) &= \\ \frac{\delta + \varepsilon}{\delta + 2\varepsilon}(x_k - \varepsilon) &+ \frac{\varepsilon}{\delta + 2\varepsilon}(x_k + \delta + \varepsilon) = x_k, \\ (1 - \lambda)(x_k - \varepsilon) + \lambda(x_{k+1} + \varepsilon) &= \\ \frac{\varepsilon}{\delta + 2\varepsilon}(x_{k+1} - \delta - \varepsilon) &+ \frac{\delta + \varepsilon}{\delta + 2\varepsilon}(x_{k+1} + \varepsilon) = x_{k+1}. \end{aligned}$$

Whence we have that

$$f(x_k) = f(x_{k+1}) \qquad (k = 0,1, \dots, N - 1),$$

thus $f(x) = f(y)$.

□

5. On the case when the set A is the set of all positive integers

Theorem 5. 1. *If $A := \mathbb{Z}_+$ and the functions f and g satisfy functional equation (1) for all $x, y \in \mathbb{Z}_+$, then these functions are in the form*

$$f(x) = \begin{cases} y_1, & \text{if } x = 2; \\ y_2, & \text{if } x = 3; \\ y_3, & \text{if } x \geqslant 4; \end{cases} \qquad g(x) = \begin{cases} y_1, & \text{if } x = 1; \\ y_2, & \text{if } x = 2; \\ y_3, & \text{if } x \geqslant 3. \end{cases} \tag{7}$$

where $y_1, y_2, y_3 \in Y$ are constants.

Proof. It is only a Hint. The proof in more details can be found in the article (Erdei et al., 2023).

If $f: \{2,3,4,\dots\} \to Y$, $g: \{1,2,3,\dots\} \to Y$ are functions satisfying equation (1) for all $x, y \in \mathbb{Z}_+$ and function the $T: \{4,5,6,\dots\} \to \mathbb{Z}_+$ is defined by

$$T(x) := \begin{cases} \dfrac{x}{2} + 3, & \text{if } x \text{ is even;} \\ \\ \dfrac{x-1}{2} + 2, & \text{if } x \text{ is odd,} \end{cases}$$

then T has the following properties:

1. $f(x) = f\big(T(x)\big)$ for all $x \geq 4$;
2. $T(x) < x$ for all $x \geq 8$;
3. If $x_0 \in \mathbb{Z}_+$, then we can give a sequence with $x_{n+1} := T(x_n)$ for all $n = 0,1,2\dots$ (this is an iteration).

Let us start the sequence from, say, 100. Thus we obtain the sequence

100, 53, 28, 17, 10, 8, 7, 5, 4, 5, 4, 5, 4…

It is easy to see that $f(x_0) = f(x_n)$ for all $x_0 \geq 4$, and $n = 0,1,2\dots$.

The behaviour of the sequence (x_n) can be seen in the following figure:

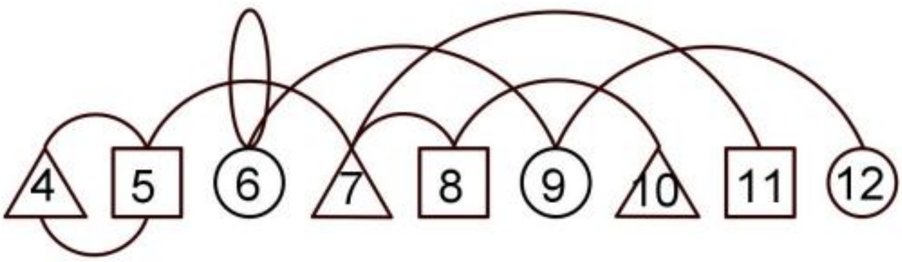

Figure 1.

Thus by a simple calculation we obtain that the function f is in the form of (1). □

Remark 5. 1. The sequence applied above is similar to the sequence in the the Collatz problem (see also 3x+1 problem).

Lothar Collatz (1910-1990) was a German mathematician who gave the following Open Problem which is the most famous problem of the 20th century.

Let us define the sequence x_n by x_0 is an arbitrary positive integer and $x_{n+1} := T(x_n)$ where

$$T(x) := \begin{cases} \dfrac{x}{2}, & \text{if } x \text{ is even;} \\ 3x+1, & \text{if } x \text{ is odd.} \end{cases}$$

Let us start the sequence from, say, 100. Thus we obtain the sequence

100, 50, 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1…

The conjecture is that wherever we start the sequence, the sequence reaches 1 in finite steps. The following Figure shows the graphe of the Collatz sequences.

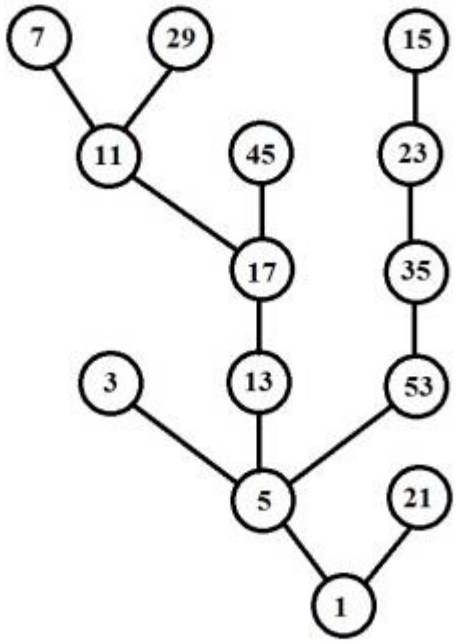

Figure 2.

6. On the case when the set A is the set of all positive dyadic rational numbers

Denote by $\mathcal{R}_{\{2\}}$ the set of all positive dyadic rational numbers, that is,

$$\mathcal{R}_{\{2\}} := \left\{ \frac{k}{2^n} \middle| \, k = 1,2,3, \dots ; n = 0,1,2, \dots \right\}. \tag{8}$$

Theorem 6. 1. *If the functions $f, g\colon \mathcal{R}_{\{2\}} \to Y$ satisfy functional equation (1) for all $x, y \in \mathcal{R}_{\{2\}}$, then these functions are constant functions.*

Proof. The Theorem is a simple consequence of the previous one. The proof in more details can be found in the article (Erdei et al., 2023). □

7. A question and the answer

The question is whether there exists a set $A \subseteq \mathbb{R}_+$ which closed under the addition and the multiplication, and the functional equation (1) on the set A has function solution (f, g) such that the ranges of this functions are infinite.

The answer is affirmative.

Theorem 7. 1. *There exists a set $A \subseteq \mathbb{R}_+$ with the following properties:*

1. *The set A is closed under addition, that is, $A + A \subseteq A$;*
2. *The set A is closed under multiplication, that is, $A \cdot A \subseteq A$;*
3. *For all set $Y = \{y_1, y_2, y_3, ...\}$ there exist functions f, $g: A \to Y$ satisfying equation (1) for all x, $y \in A$ such that $\mathcal{R}_f = Y$.*

On other word, there exists an $A \subseteq \mathbb{R}_+$ which closed under addition and multiplication, and the functional equation (1) on the set A has funtion solution (f, g) with infinite range.

Proof. Now we only give the main steps of the proof. In more details see the article (Erdei, at al., 2023).

1. Let $Y := \{y_1, y_2, y_3, ...\}$, that is, the set Y is an arbitrary infinite countable set, for example $Y = \mathbb{Q}$.
2. Denote the set of all square-free positive integers by $\mathcal{F}_2$, that is,
$$\mathcal{F}_2 = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17$$
3. Denote the sequence of the all positive prime numbers by $\mathcal{P}$, that is
$$\mathcal{P} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, ...\}.$$
4. Let $A := Q$ where the set Q is the set of all elements of the real line which can be represented in the form
$$x = \lambda_0 + \sum_{i=1}^{n} \lambda_i \sqrt{x_i}$$
where $n \in \mathbb{Z}_+$, λ_0, $\lambda_i \in \mathbb{Q}_+$ and $x_i \in \mathcal{F}_2 \setminus \{1\}$ for all $i = 1, 2, ..., n$, moreover, $x_j \neq x_k$ whenever $j \neq k$.
5. Define the function $\varphi: Q \to \mathbb{Z}_+$ by
$$\varphi\left(\lambda_0 + \sum_{k=1}^{n} \lambda_k \sqrt{x_k}\right) := \max\{k \in \mathbb{Z}_+ \mid \exists i \in \{1, \dots, n\} \exists p_k \in \mathcal{P}: p_k | x_i\}.$$
6. It is easy to see that
$$\varphi(x + y) = \varphi(xy) \qquad (x, y \in Q).$$
7. Define the functions $f: Q + Q \to Y$ and $g: Q \cdot Q \to Y$ by
$$f(u + v) := y_{\varphi(u+v)}, \qquad g(uv) := y_{\varphi(uv)} \qquad (u, v \in Q)$$
respectively.

8. Thus we obtain that the functions f, and g are solutions of functional equation (1), and the of this functions is the set Y.

□

8. Open problem

We would like to find a set $A \subseteq \mathbb{R}_+$ which closed under the addition and the multiplication, and the functional equation (1) on the set A has function solution (f, g) such that the cardinality of the ranges of this functions is continuum.

9. Summary

We give the general solution of the functional equations $f(x+y) = g(xy)$ $(x, y \in A)$ in the following cases:

1. If $A = \mathbb{R}_+$, then the functional equation can be reduced to the equation (2). The general solution of equation (1) are constant functions.
2. If $A = \mathbb{F}_+$, where $\mathbb{F}$ is an ordered field then the general solution of equation (1) are also the constant functions (Glavosits et al., 2016). The proof requires only one substitution.
3. If $A =]\alpha, \beta[\subseteq \mathbb{F}_+$ where $\mathbb{F}$ is an Archimedean ordered field, then the general solution of equation (1) are also constant functions. The proof in this case is already more complicated, among the others, special convex combinations are also needed. This case is a local version to case 2.
4. If $\mathrm{A} = \mathbb{Z}_+$, then the general solution contains three constants which are arbitrarily choice. The main idea of the proof is to use a recursively defined sequence. This sequence has a formal similarity to the series in the famous Collatz problem.
5. If the set A is the set at all positive dyadic numbers then the general solutions of equation (1) are the constant functions again. The case 5. Can be reduce to the case 4.

Although the most important aim of this article is to illustrate that similar problems may require different approaches in different environments, equation (1) also plays a role in to solve certain functional equations, for example the equation in (Glavosits et al., in preparation) , and (Glavosits et al., 2016).

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