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NEUTRAL INHOMOGENEITY IN CIRCULAR CYLINDER SUBJECTED TO PRESCRIBED BOUNDARY DISPLACEMENT

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Abstract

In this paper a single circular inhomogeneity embedded within a solid circular cylinder whose curved boundary surface is subjected to a given boundary displacement in axial direction is considered. The displacement neutrality of the coupled system of host body and inclusion is studied. The neutral inhomogeneity (inclusion) does not disturb the displacement, strain and stress fields in the host body. In this paper the deformation of the considered inhomogeneous cylinder is a linear antiplane shear deformation.

Keywords: *antiplane shear deformation, circular cylinder, elastic inclusion, neutral inhomogeneity*

1. Introduction

In the present paper the existence of neutral inhomogeneities in circular cylinder under the condition of antiplane shear deformation with prescribed boundary surface displacement is analysed. The considered solid circular cylinder is shown in *Figure 1*.

A similar problem is analysed by Benveniste and Chen in paper (Benveniste and Chen, 2003). This work deals with the Saint-Venant torsion problem when the circular bar consists of cylindrically orthotropic inclusions.

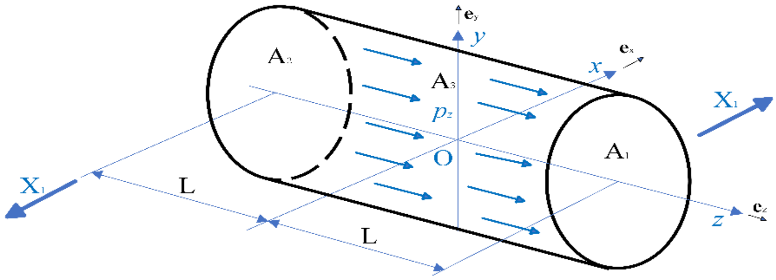

Figure 1. *The solid circular cylinder*

In our problem the solid cylinder occupies the space domain V whose boundary surfaces $\partial V = A_1 \cup A_2 \cup A_3$. On the curved boundary surface the axial displacement w is prescribed

$$w(x, y, z) = \frac{W}{R_0} x, \ (x, y, z) \in A_3 \tag{1}$$

where R_0 is the radius of the circular boundary surface A_3 *(Figure 1)*. On the boundary surface segment A_1 and A_2 the stress boundary conditions are given

$$\tau_{xz}(x, y, \pm L) = G_0 \frac{W}{R_0}, \ 0 \leq x^2 + y^2 \leq R_0, \tag{2}$$

where G_0 is the shear modulus of the material of solid circular cylinder. Under the boundary conditions (1) and (2) the deformation of elastic cylinder is antiplane shear deformation (Benveniste and Chen, 2003; Milne Thomson, 1962; Ting, 1966). The antiplane shear deformation is a special case of the state of deformation of solid body. This state is achieved when the displacements in the body are zero in the plane of interest but nonzero in the direction perpendicular to the plane. If the plane of antiplane shear deformation is the plane O_{xy} of the rectangular Cartesian frame O_{xyz} and the displacement vector $\mathbf{u}$ is represented as

$$\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z, \tag{3}$$

where $\mathbf{e}_x, \ \mathbf{e}_y, \ \mathbf{e}_z$ are the unit vectors in directions x, y and z, then the antiplane shear deformation is defined by the following equations (Milne Thomson, 1962; Ting, 1966; Barber, 2010)

$$u = 0, \ v = 0, \ w = w(x, y). \tag{4}$$

This means that if we consider a cylindrical body whose generators are parallel to axis z, all cross sections of this body have some deformations according to *Equation (4)*. The strain field of the infinitesimal antiplane shear deformation is expressed as

$$\gamma_{xz}(x, y) = \frac{\partial w}{\partial x}, \ \gamma_{yz}(x, y) = \frac{\partial w}{\partial y}, \tag{5}$$

where γ_{xz} and γ_{yz} are the shearing strain, while the other strains are zero. The cross section of the cylindrical body is a solid circle, it is denoted by A *(Figure 1)*. The boundary curve A is indicated by ∂A. It is assumed that the material of the considered host body is isotropic homogeneous and linearly elastic, its shear modulus is denoted by G_0.

The total load on the boundary surfaces A_1 and A_2 *(Figure 1)*

$$X_1 = G_0 W R_0 \pi \qquad \text{on } A_1, \tag{6}$$

$$X_2 = -G_0 W R_0 \pi \qquad \text{on } A_2. \tag{7}$$

The curved boundary surface A_3 is loaded by axial forces whose intensivity is p_z, which is

$$p_z = G_0 \frac{W}{R_0} \cos\varphi. \tag{8}$$

The resultant of the distributed force acting on the mantle of the cylindrical body vanishes

$$Z = 2L \int_0^{2\pi} G_0 \frac{W}{R_0} \cos\varphi \, \mathrm{d}\varphi = 0. \tag{9}$$

The moment about axis x, the traction acting on the boundary surface segment A_3 is zero, since

$$m_x = 2L \int_0^{2\pi} G_0 \frac{W}{R_0} \cos\varphi \, R^2 \sin\varphi \, \mathrm{d}\varphi = 2LG_0 W R_0 \int_0^{2\pi} \cos\varphi \sin\varphi \, \mathrm{d}\varphi = 0 \tag{10}$$

The moment about axis y, the traction acting on the boundary surface segment A_3 can be obtained as

$$m_y = -2L \int_0^{2\pi} p_z R_0 \cos\varphi \, R \, \mathrm{d}\varphi = -2LG_0 W R_0 \pi. \tag{11}$$

The forces X_1 and X_2 forms a couple whose moment vector is parallel to axis y and its value is *(Figure 1)*

$$\widetilde{m_y} = 2LX_1 = 2LG_0 W R_0 \pi. \tag{12}$$

The condition of equilibrium is satisfied, since

$$m_y + \widetilde{m_y} = 0. \tag{13}$$

2. Host cylinder with inclusion

The considered configuration of the host cylinder with circular inclusion is shown in *Figure 2*.

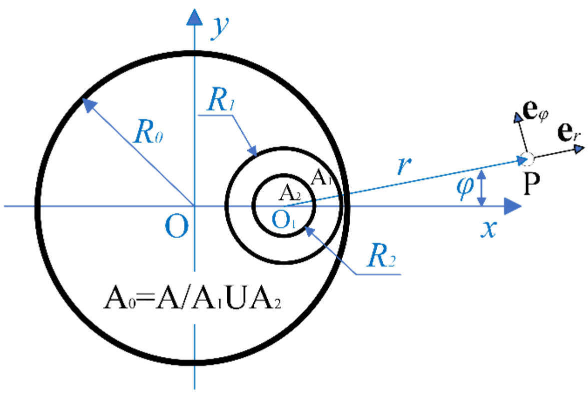

Figure 2. *The cross section of the circular cylinder with inclusion*

The inclusion consist of two parts, core and coating. The material of the core and the coating are cylindrically orthotropic, the shear moduli for the components of inclusion are G_{1rz}, $G_{1\varphi z}$ for the coating material, G_{2rz}, $G_{2\varphi z}$ for the core. A cylindrical coordinate system $O_{r\varphi z}$ is introduced. The radial coordinate is denoted by r and the polar angle is indicated by φ. The unit vectors of the cylindrical coordinate system $O_{r\varphi z}$ are $\mathbf{e}_r$, $\mathbf{e}_\varphi$ and $\mathbf{e}_z$ *(Figure 2)*. The connection between the coordinates x, y and r, φ are as follows

$$x = a + r\cos\varphi\,,\;\; y = r\sin\varphi. \tag{14}$$

The equation of the axial displacement of the host body in cylindrical coordinates is

$$w_0(r,\varphi) = \frac{W}{R_0}(a + r\cos\varphi). \tag{15}$$

The radial and circumferential stresses in cylindrical coordinates r, φ in the host body are

$$\tau_{0rz} = G_0\frac{\partial w_0}{\partial r} = G_0\frac{W}{R_0}\cos\varphi, \tag{16}$$

$$\tau_{0\varphi z} = -G_0\frac{W}{R_0}\sin\varphi. \tag{17}$$

The stress field in a cylindrical orthotropic body in case of antiplane shear deformation can be expressed as

$$\tau_{rz} = G_{rz}\frac{\partial w}{\partial r}, \qquad \tau_{\varphi z} = \frac{G_{\varphi z}}{r}\frac{\partial w}{\partial \varphi}. \tag{18}$$

The stress equilibrium equation for antiplane shear deformation is (Milne Thomson, 1962; Barber, 2010)

$$\frac{\partial\tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{1}{r}\frac{\partial\tau_{\varphi z}}{\partial\varphi} = 0, \qquad (r,\varphi) \in A. \tag{19}$$

Combination of *Equations (17)* and *(18)* gives

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{g^2}{r^2}\frac{\partial^2 w}{\partial\varphi^2} = 0, \qquad (r,\varphi) \in A, \tag{20}$$

where

$$g = \sqrt{\frac{G_{\varphi z}}{G_{rz}}}, \tag{21}$$

which is $g = 1$ for isotropic bodies.

The connections between the bodies (host body – coating, coating – core) are perfect.

3. Solution of the problem

According to *Equations (15)* and *(16)*, we look for the solution of the differential *Equation (20)* in the coating and core as

$$w_1(r,\varphi) = \frac{W}{R_0}a + (C_1 r^{g_1} + C_2 r^{-g_4})\cos\varphi\,,\;\; R_2 \le r \le R_1,\;\; 0 \le \varphi \le 2\pi, \tag{22}$$

$$w_2(r,\varphi) = \frac{W}{R_0}a + (C_3 r^{g_2} + C_4 r^{-g_2})\cos\varphi\,,\ \ 0 \le r \le R_2,\ \ 0 \le \varphi \le 2\pi, \tag{23}$$

The functions $w_1(r,\varphi)$ and $w_2(r,\varphi)$ satisfy the differential *Equation (20)*. For the bounded solution $w_2 = w_2(r,\varphi)$ we have

$$C_4 = 0. \tag{24}$$

In *Equations (22), (23)* we have

$$g_1 = \sqrt{\frac{G_{1\varphi z}}{G_{1rz}}},\ \ g_2 = \sqrt{\frac{G_{2\varphi z}}{G_{2rz}}}. \tag{25}$$

On the whole body the axial displacement and the radial shearing stress are continuous functions. From this fact it follows that

$$C_1 R_1^{g_1} + C_2 R_1^{-g_1} = \frac{W}{R_0}R_1, \tag{26}$$

$$C_1 R_1^{g_1} - C_2 R_1^{-g_1} = \frac{G_0}{G_1}\frac{W}{R_0}R_1, \tag{27}$$

$$C_1 R_2^{g_1} + C_2 R_2^{-g_1} = C_3 R_2^{g_2}, \tag{28}$$

$$C_1 R_2^{g_1} - C_2 R_2^{-g_1} = \frac{G_2}{G_1}C_3 R_2^{g_2}, \tag{29}$$

where

$$G_1 = \sqrt{G_{1rz}G_{1\varphi z}}, \qquad G_2 = \sqrt{G_{2rz}G_{2\varphi z}}. \tag{30}$$

The solution of the system of *Equations (26), (27)* for C_1 and C_2 are as follows

$$C_1 = \frac{W}{2R_0}R_1^{1-g_1}\left(1 + \frac{G_0}{G_1}\right), \tag{31}$$

$$C_2 = \frac{W}{2R_0}R_1^{1+g_1}\left(1 - \frac{G_0}{G_1}\right). \tag{32}$$

Form *Equations (28), (31)* and *(32)* C_3 can be obtained as

$$C_3 = \frac{W}{2R_0}\left[R_1^{1-g_1}R_2^{g_1-g_2}\left(1 + \frac{G_0}{G_1}\right) + R_1^{1+g_1}R_2^{-g_1-g_2}\left(1 - \frac{G_0}{G_1}\right)\right]. \tag{33}$$

From *Equation (30)* it follows that

$$C_3 = \frac{G_1 W}{2G_2 R_0}\left[R_1^{1-g_1}R_2^{g_1-g_2}\left(1 + \frac{G_0}{G_1}\right) - R_1^{1+g_1}R_2^{-g_1-g_2}\left(1 - \frac{G_0}{G_1}\right)\right]. \tag{34}$$

It is very easy to show that the *Equations (33)* and *(34)* give the same value for C_3 if

$$G_0 = G_1 = G_2, \tag{35}$$

and in this case

$$C_1 = \frac{W}{R_0}R_1^{1-g_1}, \quad C_2 = 0, \quad C_3 = \frac{W}{R_0}R_1^{1-g_1}R_2^{g_1-g_2}. \tag{36}$$

Formulae for the axial displacement and shearing stresses in the composite cylinder

$$w_0(r,\varphi) = \frac{W}{R_0}(a + r\cos\varphi\,), \tag{37}$$

$$\tau_{0rz} = \frac{G_0 W}{R_0}\cos\varphi, \tag{38}$$

$$\tau_{0\varphi z} = -\frac{G_0 W}{R_0}\sin\varphi, \tag{39}$$

$$w_1(r,\varphi) = \frac{W}{R_0}a + C_1 r^{g_1}\cos\varphi\,, \tag{40}$$

$$\tau_{1rz} = G_1 C_1 r^{g_1-1}\cos\varphi, \tag{41}$$

$$\tau_{1\varphi z} = -G_{1\varphi} C_1 r^{g_1-1}\sin\varphi, \tag{42}$$

$$w_2(r,\varphi) = \frac{W}{R_0}a + C_3 r^{g_2}\cos\varphi\,, \tag{43}$$

$$\tau_{2rz} = G_2 C_3 r^{g_2-1}\cos\varphi, \tag{44}$$

$$\tau_{2\varphi z} = -G_{2\varphi} C_3 r^{g_2-1}\sin\varphi. \tag{45}$$

The equation of the boundary curve of the cross section in cylindrical coordinate is

$$R(\varphi) = -a\cos\varphi + \sqrt{R_0^2 - a^2\sin^2\varphi}, \quad 0 \le \varphi \le 2\pi. \tag{46}$$

The axial displacement $w(r,\varphi)$ for the whole cross section is represented as

$$w(r,\varphi) = [h(r) - h(r-R_2)]w_2(r,\varphi) + [h(r-R_2) - h(r-R_1)]w_1(r,\varphi) + h(r-R_1)w_0(r,\varphi),$$
$$0 \le r \le R(\varphi),\ 0 \le \varphi \le 2\pi. \tag{47}$$

The shearing stresses τ_{rz} and $\tau_{\varphi z}$ can be obtained from the following formulae

$$\tau_{rz}(r,\varphi) = [h(r) - h(r-R_2)]\tau_{2rz}(r,\varphi) + [h(r-R_2) - h(r-R_1)]\tau_{1rz}(r,\varphi) +$$
$$+h(r-R_1)\tau_{0rz}(r,\varphi),\ 0 \le r \le R(\varphi),\ 0 \le \varphi \le 2\pi. \tag{48}$$

$$\tau_{\varphi z}(r,\varphi) = [h(r) - h(r-R_2)]\tau_{2\varphi z}(r,\varphi) + [h(r-R_2) - h(r-R_1)]\tau_{1\varphi z}(r,\varphi) +$$
$$+h(r-R_1)\tau_{0\varphi z}(r,\varphi),\ 0 \le r \le R(\varphi),\ 0 \le \varphi \le 2\pi. \tag{49}$$

4. Numerical examples

The following data are used in the presented numerical example:

$$R_0 = 0.04\text{ m}, a = 0.02\text{ m}, R_1 = 0.01\text{ m}, R_2 = 0.005\text{ m}, G_{1\varphi} = 6\cdot 10^9\text{ Pa},$$
$$G_{1r} = 8\cdot 10^{10}\text{ Pa}, G_{2r} = 6\cdot 10^9\text{ Pa}, G_{2\varphi} = 8\cdot 10^{10}\text{ Pa}, W = 0.0055\text{ m}.$$

In this case $G_0 = G_1 = G_2 = 2.19089\cdot 10^{10}$ Pa. *Figures 3, 4,* and *5* show the plots of axial displacement and of shearing stresses for $\varphi = \frac{\pi}{5}$.

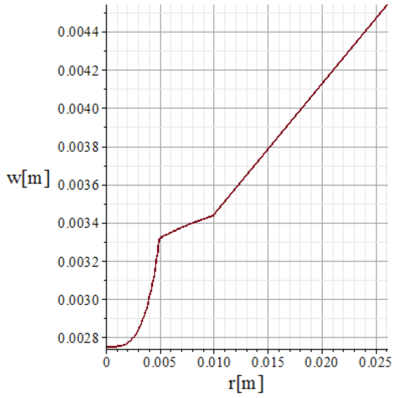

Figure 3. *Plot of the axial displacement for* $\varphi = \frac{\pi}{5}$

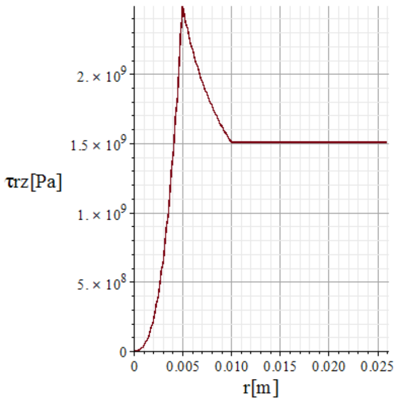

Figure 4. *Plot of shearing stress* τ_{rz} *for* $\varphi = \frac{\pi}{5}$

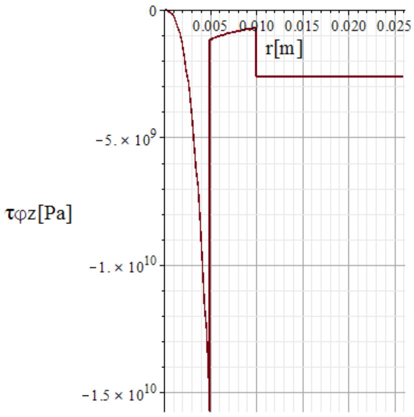

Figure 5. *Plot of shearing stress* $\tau_{\varphi z}$ *for* $\varphi = \frac{\pi}{5}$

The contour lines of the axial displacement $w(r,\varphi)$ and shearing stresses $\tau_{rz}(r,\varphi)$ are presented in *Figures 6* and *7*.

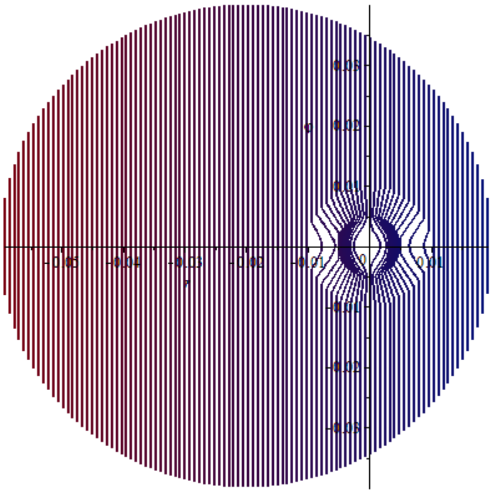

Figure 6. *The contour lines of* $w(r,\varphi)$

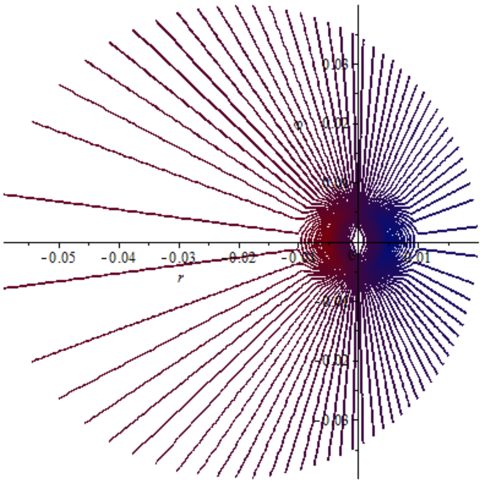

Figure 7. *The contour lines of* $\tau_{rz} = \tau_{rz}(r,\varphi)$

5. Conclusions

The problem of a single circular elastic inhomogeneity embedded within a homogeneous circular cylinder whose boundary curve has a prescribed axial displacement is investigated. The equilibrium condition of the cylinder with finite length is analysed. Our paper formulates the existence of neutral inhomogeneity if the material of neutral inhomogeneity is cylindrically orthotropic. To solve the problem, the equations of antiplane shear deformation are used.

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