APPLICATION OF A DECOMPOSITION METHOD TO FUNCTIONAL EQUATIONS

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Abstract
In this paper a decomposition theorem to functional equations is shown. As an application of this theorem the two times continuously differentiable solution of the functional equation

$$G_1\big(x(x+y)\big) + F_1(y) = G_2\big(y(x+y)\big) + F_2(y)$$

can be given with unknown functions $G_i, F_i: \mathbb{R}_+ \to \mathbb{R}$ $(i = 1,2)$ *where the equation is fulfilled for all* $x, y \in \mathbb{R}_+$ *(where* $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$*).*

Keywords: *functional equations, additive and logarithmic functions, linear differential equations, measurable solutions*

1. Introduction

The main purpose of this paper is to show the decomposition of Pexider functional equations. The Pexider equations are functional equations contained more than one unknown functions. For example, consider the well-known Lobachevsky functional equation

$$f(x+y)f(x-y) = f^2(x),$$

where the unknown function $f: \mathbb{R} \to \mathbb{R}$ satisfies the Equation for all $x, y \in \mathbb{R}$.

The Pexider version of this Equation is

$$F(x+y)G(x-y) = H^2(x),$$

where the unknown functions F, G, $H: \mathbb{R} \to \mathbb{R}$ satisfy the Equation for all $x, y \in \mathbb{R}$.

The decomposition of functional equations is a method for Pexider equations with certain symmetric structure. Such an equation results in two other, simpler equations. The general (twice continuously differentiable) solution of the original function equation can be expressed by the general (twice continuously differentiable) solutions of the obtained equations.

In section 2 the Decomposition Theorem is given. In the rest of the paper any applications of the Decomposition Theorem can be found.

1.1. The Decomposition Theorems

Theorem 1. *Let X be a set; $\circ$ and $*$ be binary operations on the set X; $(Y,+)$ be a uniquely two divisible Abelian group.*

1) *If the functions $G_i, F_i: X \to Y$ $(i = 1,2)$ are solutions of the functional equation*

$$G_1(x \circ y) + F_1(x * y) = G_2(y \circ x) + F_2(y * x) \qquad (x, y \in X), \tag{1}$$

and the functions $g, \gamma, f, \varphi: X \to Y$ are defined by

$$g := \frac{1}{2}(G_1 + G_2), \qquad f := \frac{1}{2}(F_1 + F_2),$$
$$\gamma := \frac{1}{2}(G_1 - G_2), \qquad \varphi := \frac{1}{2}(F_1 - F_2),$$

then this functions satisfy the following equations:

$$\begin{aligned} g(x \circ y) + f(x * y) &= g(y \circ x) + f(y * x), \\ \gamma(x \circ y) + \gamma(y \circ x) &= -\varphi(x * y) - \varphi(y * x) \end{aligned} \qquad (x, y \in X). \tag{2}$$

2) *If the functions $g, \gamma, f, \varphi: X \to Y$ are solutions of Equations (2), and the functions $G_i, F_i: X \to Y$ are defined by*

$$G_1 := g + \gamma, \qquad F_1 := f + \varphi,$$
$$G_2 := g - \gamma, \qquad F_2 := f - \varphi,$$

then this functions satisfy Equation (1).

The following Theorem is a special case of Theorem 1 with $x * y := y$ for all $x, y \in X$.

Theorem 2. *Let X be a set $\circ$ be a binary operation on the set X, $(Y,+)$ be a uniquely two divisible Abelian group. The functions $G_i, F_i: X \to Y$ $(i = 1,2)$ are solutions of the functional equation*

$$G_1(x \circ y) + F_1(y) = G_2(y \circ x) + F_2(x) \qquad (x, y \in X)$$

if and only if they are of the form

$$\begin{aligned} G_1(x) &= g(x) + \gamma(x), & F_1(x) &= f(x) - \gamma(x \circ x), \\ G_2(x) &= g(x) - \gamma(x), & F_2(x) &= f(x) + \gamma(x \circ x) \end{aligned}$$

for all $x \in X$ where the functions $f, g, \gamma: X \to Y$ are solutions of the equations

$$\begin{aligned} g(x \circ y) + f(y) &= g(y \circ x) + f(x) && (x, y \in X) \\ \gamma(x \circ x) + \gamma(y \circ y) &= \gamma(x \circ y) + \gamma(y \circ x) && (x, y \in X). \end{aligned}$$

We investigate the case

$$X := \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}, \; x \circ y := x(x + y)$$

for all $x, y \in \mathbb{R}_+$, $Y(+) := \mathbb{R}(+)$, and we want to find the twice continuously differentiable solutions. By Theorem 2 instead of equation

$$G_1\big(x(x+y)\big)+F_1(y)=G_2\big(y(x+y)\big)+F_2(x) \qquad (x,y\in\mathbb{R}_+) \tag{3}$$

with unknown functions $G_i, F_i\colon \mathbb{R}_+\to\mathbb{R}$ $(i=1,2)$ where the equation is fulfilled for all $x,y\in\mathbb{R}_+$ it is enough to give the twice continuously differentiable solutions of the equations

$$g\big(x(x+y)\big)+f(y)=g\big(y(x+y)\big)+f(y) \qquad (x,y\in\mathbb{R}_+) \tag{4}$$

and

$$\gamma\big(x(x+y)\big)+\gamma\big(y(x+y)\big)=\gamma(2x^2)+\gamma(2y^2) \qquad (x,y\in\mathbb{R}_+). \tag{5}$$

In equation (5) we will use the function $\delta\colon\mathbb{R}_+\to\mathbb{R}$ defined by $\delta(x):=\gamma(2x^2)$ for all $x\in\mathbb{R}_+$.

Equation (5) (and with it its Pexider version, equation (3) is related to equations containing means and equations containing the Gauss composition of these means see (Borwein et al., 1998; Daróczy, 2007) and (Daróczy et al., 1977, 2002, 2007a, 2007b), so these equations are very important.

The measurable solutions of Equation of (4) can be easily obtained by using the method of (Járai, 1979 and 2005), but we do not deal with finding the measurable solutions, because we want to know the general solutions in more general settings see our conjecture in the last section of this paper.

2. The twice continuously differentiable solutions of Equation (4)

In his paper (Narumi, 1923) was the first to use differential calculus to solve functional equations. Nowadays, there are many different methods for constructing differential operators that can be used to reduce a functional equation to a differential equation (Gilányi, 1998; Házy, 2004) and (Páles, 1992). Instead of them we will use only two similar and natural differential operators.

Proposition 1. *If the functions $F_i, G_i\colon\mathbb{R}_+\to\mathbb{R}$ are twice differentiable solutions of equation (3) then*

$$0=\tfrac{-2x+y}{x+2y}F_1{}'(y)+\tfrac{-2x-y}{x+2y}F_1{}''(y)+\tfrac{-x+2y}{x+2y}F_2{}'(x)+\tfrac{x}{y}F_2{}''(x)$$

for all $x,y\in\mathbb{R}_+$.

Proposition 2. *If the functions $f, g\colon\mathbb{R}_+\to\mathbb{R}$ are twice differentiable solutions of the equation (4), and the constants c_1, c_2; and functions $a_0, a_1, \Phi\colon\mathbb{R}_+\to\mathbb{R}$ are defined by*

$$\begin{aligned} c_1 &:= f'(1), \quad c_2:=f''(1),\\ a_0(x) &:= c_1\frac{2x-1}{x(x+2)}+c_2\frac{2x+1}{x(x+2)} \qquad (x\in\mathbb{R}_+)\\ a_1(x) &:= \frac{x-2}{x(x+2)} \qquad (x\in\mathbb{R}_+),\\ \Phi(x) &:= f'(x) \qquad (x\in\mathbb{R}_+), \end{aligned}$$

then

$$\Phi'(x)=a_0(x)+a_1(x)\Phi(x) \qquad (x\in\mathbb{R}_+) \tag{6}$$

For to solve the above differential equation we can apply the well-known formula

$$\Phi(x)=\exp\left(\int_1^x a_1(u)du\right)\left(c_1+\int_1^x\frac{a_0(v)}{\exp\left(\int_1^v a_1(u)du\right)}dv\right) \qquad (x\in\mathbb{R}_+).$$

Proposition 3. *Preserve the notation of Proposition 2. The solution of differential equation (6) (with boundary values* $\Phi(1) = c_1$, $\Phi'(1) = c_2$*) is*

$$\Phi(x) = c_1 \frac{x^2+1}{2x} + c_2 \frac{x^2-1}{2x} \qquad (x \in \mathbb{R}_+).$$

Theorem 3. *The twice continuously differentiable solution of equation (4) is*

$$\begin{aligned} g(x) &= C_1 x + C_2 \ln(x) + C_4 \qquad (x \in \mathbb{R}_+), \\ f(x) &= C_1 x^2 + C_2 \ln(x) + C_3 \qquad (x \in \mathbb{R}_+) \end{aligned}$$

where C_i *are arbitrary constants for* $i = 1,2,3,4$.

3. The twice continuously differentiable solutions of equation (3) and (5)

Proposition 4. *If the function* $\gamma: \mathbb{R}_+ \to \mathbb{R}$ *is twice differentiable solutions of Equation (3), and the functions* $b_0, b_1, \Phi: \mathbb{R}_+ \to \mathbb{R}$ *and the constants* d_1, d_2 *are defined by*

$$\begin{aligned} \delta(x) &:= \gamma(2x^2), \\ d_1 &:= \delta'(1), \quad d_2 := \delta''(1), \\ b_0(x) &:= d_1 \frac{2x-1}{x(x+2)} + d_2 \frac{2x+1}{x(x+2)} \qquad (x \in \mathbb{R}_+) \\ b_1(x) &:= \frac{x-2}{x(x+2)} \qquad (x \in \mathbb{R}_+), \\ \Psi(x) &:= \delta'(x) \qquad (x \in \mathbb{R}_+), \end{aligned}$$

then

$$\Psi'(x) = b_0(x) + b_1(x)\Phi(x) \qquad (x \in \mathbb{R}_+)$$

Proposition 5. *Preserve the notation of Proposition 4. The solution of differential equation (6) is*

$$\Psi(x) = d_1 \frac{-5x^2 + 16x + 7}{18x} + d_2 \frac{-(x-1)^2}{2x} \qquad (x \in \mathbb{R}_+).$$

Theorem 4. *If the function* $\gamma: \mathbb{R}_+ \to \mathbb{R}$ *is a twice continuously differentiable solution of Equation (4), then it is a constant function.*

Now, we can give the twice continuously differentiable solution of Equation (2).

Theorem 5. *The twice continuously differentiable solution of Equation (2) is*

$$\begin{aligned} G_1(x) &= C_1 x + C_2 \ln(x) + D_1, \qquad & F_1(x) &= C_1 x^2 + C_2 \ln(x) + D_2, \\ G_2(x) &= C_1 x + C_2 \ln(x) + D_3, \qquad & F_2(x) &= C_1 x^2 + C_2 \ln(x) + D_4 \end{aligned}$$

for all $x \in \mathbb{R}_+$ *where* C_i $i = 1,2$, *and* $D_j \in \mathbb{R}$ $j = 1,2,3,4$ *are arbitrary constants such that* $D_1 + D_2 = D_3 + D_4$.

4. Additional applications and conjectures

In this chapter we shell apply the notations of Theorem 2. In paper (Glavosits et al., 2005) was investigated the case, when $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+,\cdot)$ is an ordered field (the operation $\cdot$ is commutative) the $\mathbb{T}_+$ is the set of positive elements of the set $\mathbb{F}$,

$$x \circ y := \frac{x}{y+1}$$

for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group. In these settings the general solution of equation (1) is

$$\begin{aligned} G_1(x) &= l_1 x + l_2(x+1) + l_3(x) + d_1, \\ G_2(x) &= l_1 x + l_2(x+1) - l_3(x) + d_3, \\ F_1(x) &= l_1\big(x(x+1)\big) + l_2(x) - l_3\left(\frac{x+1}{x}\right) + d_2, \\ F_2(x) &= l_1\big(x(x+1)\big) + l_2(x) + l_3\left(\frac{x+1}{x}\right) + d_4, \end{aligned}$$

for all $x \in \mathbb{F}_+$ where $l_i: \mathbb{F}_+ \to Y$ $(i = 1,2,3)$ are arbitrary logarithmic functions, $d_i \in Y$ $(i = 1,2,3,4)$ are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$. (Concerning the logarithmic functions see (Aczél et al., 1989) and (Kuczma, 2009). This result cannot be improved.

In paper (Glavosits et al.) was investigated the case when $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+,\cdot)$ is an Archimedean ordered field (the operation $\cdot$ is commutative),

$$x \circ y := x(y+1)$$

for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group. In this settings the general solution of Equation (1) is

$$\begin{aligned} G_1(x) &= a(x) + l_2(x) + l_3(x) + d_1, \\ G_2(x) &= a(x) + l_2(x) - l_3(x) + d_3, \\ F_1(x) &= -a(x) + l_2\left(\frac{x}{x+1}\right) - l_3\big(x(x+1)\big) + d_2, \\ F_2(x) &= -a(x) + l_2\left(\frac{x}{x+1}\right) + l_3\big(x(x+1)\big) + d_4, \end{aligned}$$

for all $x \in \mathbb{F}_+$ where $a: \mathbb{F}_+ \to Y$ is an arbitrary additive function, $l_i: \mathbb{F}_+ \to Y$ $(i = 2,3)$ are arbitrary logarithmic functions, $d_i \in Y$ $(i = 1,2,3,4)$ are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$. (Concerning the additive and logarithmic functions see (Aczél et al., 1989) and (Kuczma, 2009). This result could perhaps be improved writing simply ordered field instead of Archimedean ordered field, that is,

Conjecture 1. *If $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+,\cdot)$ is an ordered field, $x \circ y := x(y+1)$ for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group, then the general solution of Equation (1) is of the form (6) for all $x \in \mathbb{F}_+$ where $a: \mathbb{F}_+ \to Y$ is an arbitrary additive function, $l_i: \mathbb{F}_+ \to Y$ $(i = 2,3)$ are arbitrary logarithmic functions, $d_i \in Y$ $(i = 1,2,3,4)$ are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$.*

The above two problems have probability background see (Páles, 1992; Arnold et al., 1992) and (Glavosits et al., 2005). Two pairs of new equations can be derived from the above two equations, the Pexider versions of the second of these equations (which contain the unknown gamma function) characterize the logarithmic functions see (Glavosits et al., 2016).

Finally, we can formulate our Conjecture concerning the main equation of this paper.

Conjecture 2. *If $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+,\cdot)$ is an ordered field, $x \circ y := x(x + y)$ for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group, then the general solution of Equation (2) is*

$$\begin{aligned} G_1(x) &= a(x) + l(x) + d_1, & F_1(x) &= a(x^2) + l(x) + d_2, \\ G_2(x) &= a(x) + l(x) + d_3, & F_2(x) &= a(x^2) + l(x) + d_4 \end{aligned}$$

for all $x \in \mathbb{F}_+$ where $a: \mathbb{F}_+ \to Y$ is an arbitrary additive function, $l: \mathbb{F}_+ \to Y$ is an arbitrary logarithmic function, $d_i \in Y$ $i = 1,2,3,4$ are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$.

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