SOME REMARK ON THE FUNCTIONAL EQUATION

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Abstract
The appearance of function equations can be traced back to a long time ago. The oldest and perhaps the best-known function equation is the Cauchy function equation. The Cauchy function equation is as follow

$$f(x+y) = f(x) + f(y) \qquad (x, y \in R+).$$

The trivial solution of Cauchy equation is simple, but the general solution is very complex.
Later several function equations appeared, which either arose during some practical problem or became known from a theoretical perspective.
The function equation Hosszú is also such a function equation, introduced by Miklós Hosszú based on theoretical considerations. The function equation Hosszú has the form below:
We further generalize the above equation and give its solution. After the generalization, we get the following function equation.

$$f(|x+y-(x\circ y)|) + g(x\circ y) = g(x) + g(y) \qquad (x, y \in R+)$$

with unknown functions $f, g : \mathrm{R}+ := \{x \in \mathrm{R} \mid x > 0\} \to \mathrm{R}$, *and binary operation defined by* $x \circ y := \left(x^{\frac{1}{c}} + y^{\frac{1}{c}}\right)^{c}$ *for all* $x, y \in \mathrm{R}_{+}$ *where* $c \in \mathrm{R}$, $c \in \{2, 3\}$ *is a fixed constant.*

Keywords: *functional equations, additive functions, Hosszú type functional equations, Hosszú functional equation, Hosszú cycle*

1. Introduction

At the International Symposium on Functional Equation conference held on Zakopane (Poland) in 1967 the functional equation

$$f(x+y-xy) + f(xy) = f(x) + f(y) \tag{1.1}$$

where the unknown function $f\colon \mathrm{R}_{+} \to \mathrm{R}$ satisfies the equation for all $x, y \in \mathrm{R}$ was proposed to investigate in the first time by M. Hosszú. This equation is known as 1 Manuscript for the Annales Mathematicae et Informaticae June 23, 2023 Hosszú functional equation.

Z. Daróczy (Daróczy, 1969; Daróczy, 1971) D. Blanusa (Blanuša, 1970), and H. Swiatak (Światak, 1968; Światak, 1971) proved that the general solution of *equation (1.1)* is in the form $f(x) = A(x) + C$ for all $x \in$ R where A : R → R is an additive function [that is, $A(x+y) = A(x)+A(y)$ is fulfilled for all x, $y \in$ R (Aczél, 1966; Aczél and Dhombres, 1989; Kuczma, 1985)], and C is a real constant.

Since 1969, many researchers have investigated the Hosszú equation and its generalizations. In papers (Davison, 2001; Davison, 1974a; Davison, 1974b; Davison and Redlin, 1980) the *equation (1.1)* was investigated on various abstract structures. In paper (Fenyő, 1970) a Pexider version (that is functional equation with more unknown function [Aczél, 1983; Aczél and Dhombres, 1989; Kuczma, 1985; Glavosits and Lajkó, 2016]) of *equation (1.1)* and its locally integrable function solutions can be found. See also (Lajkó et al., 2009; Lajkó and Mészáros, 2015).

The general solution of equations

$$f(xy) + g(x + y - xy) = f(x) + f(y) \tag{1.2}$$

$$f(xy) + g(x + y - xy) = h(x) + h(y) \tag{1.3}$$

was given by K. Lajkó (Lajkó, 2001) in the following cases:

Problem A: If the unknown functions f, g :]0, 1[→ R satisfy the *equation (1.2)* for all $x, y \in]0, 1[$, then there exist additive functions $A1$, $A2$, and a constant $C \in$ R such that

$$f(x) = A_1(x) + A_2(\log x) + C, \ (x \in \mathrm{R}),$$

$$g(x) = A_1(x) + C, \ (x \in \mathrm{R}).$$

(where log denotes the natural logarithm function).

Problem B: If the unknown functions f, g, h : R → R satisfy the *equation (1.3)* for all x, $y \in$ R, then there exist an additive function A and constants Ci (i = 1, 2, 3) such that

$$f(x) = A(x) + C_2 \ (x \in \mathrm{R}),$$

$$g(x) = A(x) + C_3, \ (x \in \mathrm{R}),$$

$$h(x) = A(x) + C_1, \ (x \in \mathrm{R}).$$

If the unknown functions f, h : $\mathrm{R}_0 := \mathrm{R} \setminus \{0\}$ → R, and g : R → R satisfy the *equation (1.3)* for all x, $y \in$ R0, then there exist additive functions A_1, A_2 and constants C_i (i = 1, 2, 3) such that

$$f(x) = A_1(x) + A_2(\log |x|) + C_3 \ (x \in \mathrm{R}_0),$$

$$g(x) = A_1(x) + C_2, \ (x \in \mathrm{R}),$$

$$h(x) = A_1(x) + A_2(\log |x|) + C_1, \ (x \in \mathrm{R}_0).$$

If the unknown functions f : R → R, and g, h : $\mathrm{R}_1 := \mathrm{R} \setminus \{1\}$ → R satisfy the *equation (1.3)* for all x, $y \in$ R1, then there exist additive functions A_1, A_2 and constants C_i (i = 1, 2, 3) such that

$$f(x) = A_1(x) + A_2(\log |1 - x|) + C_3 \ (x \in \mathrm{R}_1),$$

$$g(x) = A_1(x) + C_2, \ (x \in \mathrm{R}).$$

$$h(x) = A_1(x) + A_2(\log|1 - x|) + C_1, (x \in R_1).$$

In his paper (Daróczy, 1999) Z. Daróczy investigate the functional equation f

$$(x + y - x \circ y) + f(x \circ y) = f(x) + f(y) \ (x, y \in R_+) \quad (1.4)$$

with unknown function $f : R \to R$ where the binary operation $\circ$ is defined by $x \circ y := \ln(e\ x + e\ y)$ for all $x, y \in R+$. The general solution of this equation is $f(x) = A(x)+C$ for all $x \in R$ where A is an additive function, C is a real constant. The main purpose of our present paper is to give the general solution of the functional equation

$$f(|x + y - (x \circ y)|) + g(x \circ y) = g(x) + g(y) \ (x, y \in R_+) \quad (1.5)$$

with unknown functions f, $g : R+ \to R$. The binary operation is defined by

$$x \circ y := (x\ 1\ c + y\ 1\ c\ )\ c\ (x, y \in R_+), \quad (1.6)$$

where $c \in R \setminus \{0, 1\}$ is a fixed constant. We also consider the functional equation

$$f(|x + y - (x \circ y)|) + g(x \circ y) = h(x) + h(y) \ (x, y \in R_+) \quad (1.7)$$

with unknown functions f, g, $h : R_+ \to R$ and binary operation $\circ$ is defined by (1.6). The *equation (1.5)* is a common generalization of *equations (1.2)* and *(1.4)*. The *equation (1.7)* is a common generalization of *equations (1.3)* and *(1.4)*. The devices needed to the problems we set out, and to the earlier problems are the theorems giving the solutions of the restricted Pexider additive functional equations (in the rest briefly Additive Extension Theorems) and the application of the Hosszú cycle. Let $D \subseteq R^2$ be a non-empty connected set. Define the sets

$$Dx := \{u \in R \mid \exists v \in R : (u, v) \in D\},$$

$$Dy := \{v \in R \mid \exists u \in R : (u, v) \in D\},$$

$$Dx{+}y := \{z \in R \mid \exists (u, v) \in D : z = u + v\}.$$

The functional equation f

$$(x + y) = g(x) + h(y) \ ((x, y) \in D)$$

with unknown functions $f : D_{x+y} \to R$, $g : Dx \to R$, $h : Dy \to R$ is a restricted Pexider additive functional equation. According to Rimán's Extension Theorem (Rimán, 1976), there exist an additive function A : $R \to R$ and constants C_i ($i = 1, 2$) such that

$$f(u) = A(u) + C_1 + C_2 \ (u \in D_{x+y}),$$

$$g(v) = A(v) + C_1 \ (v \in D_x),$$

$$h(z) = A(z) + C_2 \ (z \in D_y).$$

Concerning the Additive Extension Theorems see also (Aczél, 1983; Daróczy and Losonczi, 1967; Kuczma, 1985; Radó and Baker, 1987; Glavosits and Karácsony, 2023; Glavosits, in preparation). A Hosszú cycle is a functional equation

$$F(x + y, z) + F(x, y) = F(x, y + z) + F(y, z) \ (x, y, z \in D)$$

with unknown function $F : A_2 \to D$, where $A = A(+)$ is a semi-group, $D = D(+)$ is an Abelian semi-group. The first appearance of this equation was in (Hosszú, 1971), although many researchers use the Hosszú Cycle to solve functional equations for example *Equations (1.1)*, and *(1.2)*.

2. The decomposition of *equation (1.5)* by Hosszú cycle

Theorem 2.1. If the functions $f, g : R_+ \to R$ satisfy the functional *Equation (1.5)* where the binary operation $\circ$ is defined by (1.6), then f satisfies the functional equation

$$f(|(x+y+z)^c - ((x+y)^c + z^c)|) + f(|(x+y)^c - ((x^c + y^c)|) = f(|(x+y+z)^c - (x^c + (y+z)^c)|) + f(|(y+z)^c - (y^c + z^c)|) \quad (x, y \in R_+). \tag{2.1}$$

Proof. For the proof we shall use the well-known Hosszú Cycle (see Daróczy, 1999). Define the function $F : R\ 2 + \to R$ by

$$F(x, y) := g(x) + g(y) - g(x \circ y) \quad (x, y \in R_+).$$

Since the operation $\circ$ is associated thus we have that

$$F(x \circ y, z) + F(x, y) = F(x, y \circ z) + F(y, z) \quad (x, y, z \in R_+) \tag{2.2}$$

By *equation (1.5)* we have that

$$F(x, y) = f(|x + y - (x \circ y)|) \quad (x, y \in R_+). \tag{2.3}$$

From *equations (2.2)* and *(2.3)* we have the *equation (2.1)*.
Although the following proposition is well known, we give a short proof for the sake of the readers.

Proposition 2.2. Let $c \in R \setminus \{0, 1\}$ and define the function

$$\phi c : R_+ \to R_+ \text{ by } \phi c(x) := x^c \quad (x \in R_+).$$

a If $c > 1$, then the function ϕc is strictly superadditive in the sense that

$$(x + y)^c > x^c + y^c \quad (x, y \in R_+) \tag{2.4}$$

b If $0 < c < 1$, or $c < 0$ then the function ϕc is strictly subadditive in the sense that

$$x^c + y^c > (x + y)^c \quad (x, y \in R_+) \tag{2.5}$$

Proof. Let $c \in R \setminus \{0, 1\}$ be a fixed constant and investigate the functions $\psi c : R+ \to R+$, and $\delta c : R+ \to R$ by

$$\psi_c(x) := \frac{1 + x^c}{(1+x)^c} \quad (x \in R_+),$$

$$\delta_c(x) := \frac{1}{|c|}\frac{d}{dx}\log(\psi_c(x)) = sgn(c)\frac{x^{c-1} - 1}{(1+x)^c(1+x)} \quad (x \in R_+).$$

Case 1. Let $0 < c < 1$. Since the function $\delta_c > 0$ on the interval $]0, 1[$, and $\delta_c < 0$ on the interval $]1, +\infty[$ thus the function ψ_c is increasing in the interval $]0, 1[$, and it is decreasing on the interval $]1, +\infty[$, moreover,

$$\lim_{x\to 0+0} \psi_c(x) \lim_{x\to\infty} \psi_c(x) = 1,$$

whence we have that $\psi_c(x) > 1$ for all $x \in R_+$, that is

$$1 + z^c > (1 + z)^c \ (z \in R_+). \tag{2.6}$$

Take the substitution $z \leftarrow \frac{y}{x}$ for all x, $y \in R_+$ in inequalities (2.6) then multiply both sides of the obtained inequality by x^c whence we obtain that the function ϕ_c is subadditive.

Case 2. Let $c > 1$. Since $0 < \frac{1}{c} < 1$, and by step 1 the function $\phi_{\frac{1}{c}}$ is strictly subadditive, and strictly increasing thus the function $\phi_c = \left(\phi_{\frac{1}{c}}\right)^{-1}$ is superadditive.

Case 3. Let $-1 \leqslant c < 0$. It is easy to see that the function $\delta_c < 0$ on the interval $]0, 1[$, and $\delta_c > 0$ on the interval $]1, +\infty[$ whence we have that the function ψc is decreasing on the interval $]0, 1[$ and it is increasing on the interval $]1, +\infty[$. Since $\phi_c(1) > 1$ thus $\phi_c(x) > 1$ for all $x \in R_+$ thus the inequality (2.6) is satisfied again so the function ϕ_c is subadditive again.

Case 4. Let $-\infty < c \leqslant -1$. Since $-1 \leqslant \frac{1}{c} < 0$ by step 3 the function $\phi_{\frac{1}{c}}$ is strictly subadditive, and strictly decreasing thus the function $\phi_c = \left(\phi_{\frac{1}{c}}\right)^{-1}$ is subadditive again.

Example 2.3. Since $e^x = \sum x/n!$ thus by Proposition 2.2 it is easy to see that $e^{x+y} + 1 > e^x + e^y$ for all x, $y \in R_+$, and $\log(x + 1) + \log(y + 1) > \log(x + y + 1)$ for all x, $y \in R_+$. It is also easy to see that $\log(x + y) > \log(x) + \log(y)$ for all x, $y \in]0, 1[$. Define the function $\phi : R_+ \to R$ by $\phi(x) := x + x^2 + x^3$ for all $x \in R_+$. Then function ϕ is a strictly superadditive bijection.

Our references concerning subadditive, and superadditive functions is (Nikodem et al., 2000).

Corollary 2.4. Preserving the notations of Theorem 2.1 by Proposition 2.2 it is easy to see that

• if $c > 1$, then the function f satisfies the equation

$$\begin{aligned} f((x + y + z)^c - ((x + y)^c + z^c)) + f((x + y)^c - ((x^c + y^c)) \\ = f((x + y + z)^c - (x^c + (y + z)^c)) \\ + f((y + z)^c - (y^c + z^c)) \ (x, y, z \in R_+). \end{aligned} \tag{2.7}$$

• if $0 < c < 1$ or $c < 0$, then the function f satisfies the equation

$$\begin{aligned} f((x + y)^c + z^c - (x + y + z)^c) + f((x^c + y^c - (x + y)^c) \\ = f(x^c + (y + z)^c - (x + y + z)^c) \\ + f(y^c + z^c - (y + z)^c) \ (x, y, z \in R_+). \end{aligned} \tag{2.8}$$

Corollary 2.5. If the function $f : R_+ \to R$ satisfy the functional *Equation (2.7)* or *Equation (2.8)* for all $x, y, z \in R_+$ thus it is also satisfies the equation

$$f\left(\frac{(x+y+z)^c-((x+y)^c+z^c)}{(x+y+z)^c-(x^c+(y+z)^c)}\right)+f\left(\frac{(x+y)^c-(x^c+y^c)}{(x+y+z)^c-(x^c+(y+z)^c)}\right)$$
$$=f(1)+f\left(\frac{(y+z)^c-(y^c+z^c)}{(x+y+z)^c-(x^c+(y+z)^c)}\right)$$
$$(x, y, z \in \mathrm{R}+). \tag{2.9}$$

3. On the *equation (2.9)* in the cases c = 2,3

Proposition 3.1. If the function $f : \mathrm{R}_+ \to \mathrm{R}$ satisfies the equation

$$f(u) + f(v) = f(1) + f(u + v - 1) \ ((u, v) \in D) \tag{3.1}$$

where the set $D \subseteq \mathrm{R}^2$ is defined by

$$D := \{(u, v) \in \mathrm{R}^2 \mid v > -u + 1, v < 1\}, \tag{3.2}$$

then there exists an additive function $A : \mathrm{R} \to \mathrm{R}$ and a constant $C \in \mathrm{R}$ such that.

$$f(x) = A(x) + C \ (x \in \mathrm{R}_+).$$

Proof. Define the function $g :]1, \infty[\to \mathrm{R}$ by

$$g(z) = f(z - 1) + f(1) \ (z \in]1, \infty[).$$

Then the functions f and g satisfy the equation

$$g(u + v) = f(u) + f(v) \ ((u, v) \in D)$$

where the set D is defined by (3.2). Thus by Rimán's Extension Theorem (see Rimán, 1976; Glavosits, in preparation) we obtain that there exist an additive function $A : \mathrm{R} \to \mathrm{R}$ and a constant $C \in \mathrm{R}$ with

$$f(u) = A(u) + C \ (u \in Dx = \mathrm{R}_+).$$

Theorem 3.2. If the function $f : \mathrm{R}_+ \to \mathrm{R}$ satisfy *equation (2.9)* for all $x, y, z \in \mathrm{R}_+$ with constant $c \in \{2, 3\}$, there exist an additive function $A_1 : \mathrm{R} \to \mathrm{R}$ and a constant $C_1 \in \mathrm{R}$ such that

$$f(x) = A_1(x) + C_1 \ (x \in \mathrm{R}_+). \tag{3.3}$$

Proof. Case 1. Let $c = 2$. By *equation 2.9* the function f satisfies the equation

$$f\left(\tfrac{(x+y)z}{x(y+z)}\right)+f\left(\tfrac{y}{y+z}\right)=f(1)+f\left(\tfrac{yz}{x(y+z)}\right) \ (x, y, z \in \mathrm{R}_+). \tag{3.4}$$

Take the substitution in *Equation (3.4)*

$$y \leftarrow \frac{x(u+v-1)}{1-v} \qquad z \leftarrow \frac{x(u+v-1)}{v}$$

thus we have that the function f satisfies the *equation (3.1)* where the set $D \subseteq \mathrm{R}^2$ is defined by (3.2). By Proposition 3.1 we have that the function f is in the form of (3.3).

Case 2. Let c = 3. By *equation (2.9)* the function f satisfies the equation

$$f\left(\frac{(x+y)z}{x(y+z)}\right)+f\left(\frac{y(x+y)}{(y+z)(x+y+z)}\right)=f(1)+f\left(\frac{yz}{x(x+y+z)}\right)\ (x,y,z\in \mathrm{R}+). \tag{3.5}$$

Take the substitution in *Equation (3.5)*

$$y\leftarrow\frac{\sqrt{uvx^2(u+v-1)+x(u+v-1)}}{1-v}\quad z\leftarrow\frac{x(u+v-1)}{v}$$

whence we have that the function f satisfy the *equation (3.1)* where the set D is defined by (3.2). The rest of the proof of this case is analogous to the case c = 3.

4. Results and open problems

Theorem 4.1. If $\phi : \mathrm{R}_+ \to \mathrm{R}_+$ is a strictly superadditive bijection, the binary operation ∘ is defined by

$$x\circ y := \phi(\phi -1\ (x) + \phi -1\ (y))\ (x, y \in \mathrm{R}_+), \tag{4.1}$$

$f : \mathrm{R}_+ \to \mathrm{R}$ is a function such that that exist an additive function $A1 : \mathrm{R}_+ \to \mathrm{R}$ and a constant $C \in \mathrm{R}$ with

$$f(x) = A_1(x) + C\ (x \in \mathrm{R}_+), \tag{4.2}$$

$g, h : \mathrm{R}_+ \to \mathrm{R}$ are functions such that

$$f(|x + y - (x\circ y)|) + g(x\circ y) = h(x) + h(y)\ (x, y \in \mathrm{R}_+), \tag{4.3}$$

then there exist an additive function $A_2 : \mathrm{R} \to \mathrm{R}$ and constant C_2 with

$$g(x) = -A_1(x) + A_2(\phi^{-1}(x)) + 2C_2 - C_1\ (x \in\mathrm{R}_+),$$

$$h(x) = -A_1(x) + A_2(\phi^{-1}(x)) + C_2\ (x \in\mathrm{R}_+). \tag{4.4}$$

Proof. During the proof, ∘ will denote the usual function composition (for example $(g\circ\phi)(x) := g(\phi(x))$). Since $\phi(x + y) > \phi(x) + \phi(y)$ thus from *equation (4.3)* we have that

$$f(\phi(x + y) - (\phi(x) + \phi(y))) + g(\phi(x + y)) = h(\phi(x)) + h(\phi(x))\ (x, y \in \mathrm{R}_+). \tag{4.5}$$

From *Equations (4.2)* and *(4.5)* we have that

$$((A_1\circ\phi) + (g\circ\phi) + C_1))(x + y) = ((A_1\circ\phi) + (h\circ\phi))(x) + ((A_1\circ\phi) + (h\circ\phi))(y)\ (x, y \in \mathrm{R}_+). \tag{4.6}$$

Define de functions $F, G : \mathrm{R}_+ \to \mathrm{R}$ by

q

$$F(x) : = ((A_1\circ\phi) + (g\circ\phi) + C1)(x)\ (x \in\mathrm{R}_+), \tag{4.7}$$

$$G(x) : = ((A_1\circ\phi) + (h\circ\phi))(x)\ (x \in\mathrm{R}_+). \tag{4.8}$$

From *equation (4.6)* we have that

$$F(x + y) = G(x) + G(y)\ (x, y \in \mathrm{R}_+) \tag{4.9}$$

From *equation (4.9)* by Rimán's Extension Theorem we have that there exist an additive function $A2 : \mathrm{R} \to \mathrm{R}$ and a constant $C2 \in \mathrm{R}$ such that

$$F(x) = A_2(x) + 2C_2\ (x \in\mathrm{R}_+), \tag{4.10}$$

$$G(x) = A_2(x) + C_2 \quad (x \in \mathrm{R}+). \tag{4.11}$$

From *equations (4.8),* and *(4.11)* we obtain *equation (4.4).*

Theorem 4.2. Let $\phi : \mathrm{R}_+ \to \mathrm{R}_+$ be a strictly subadditive bijection. Preserving the notation of Theorem 4.1 the functions g, h is in the form g

$$(x) = A_1(x) + A_2(\phi^{-1}(x)) + 2C_2 - C_1 \quad (x \in \mathrm{R}_+), \tag{4.12}$$

$$h(x) = A_1(x) + A_2(\phi^{-1}(x)) + C_2 \quad (x \in \mathrm{R}_+). \tag{4.13}$$

Proof. The proof is analogous to the proof of Theorem 4.1.

Theorem 4.3. If the functions $f, g : \mathrm{R}+ \to \mathrm{R}$ satisfy the functional *Equation (1.5)* where the operation $\circ$ is defined by (1.6) where $c \in \{2, 3\}$, then there exist an additive functions $A_1, A_2 : \mathrm{R} \to \mathrm{R}$ and a constant $C \in \mathrm{R}$ such that

$$f(x) = A_1(x) + C \quad (x \in \mathrm{R}_+), \tag{4.14}$$

$$g(x) = A_1(x) + A_2\left(\frac{1}{x^c}\right) + C \quad (x \in \mathrm{R}_+). \tag{4.15}$$

Proof. The proof is evident by Theorem 2.1, Corollary 2.4, Corollary 2.5, Theorem 3.2, and Theorem 4.1.

Theorem 4.1, Theorem 4.2, and Theorem 4.3 suggest the following open problems.

Problem A: Does Theorem 4.3 hold for all $c \in \mathrm{R}_+ \setminus \{0, 1\}$?

Our conjecture is that it remains true.

Problem B: Find the general solution of *Equation (1.7)* with unknown functions $f, g, h : \mathrm{R}_+ \to \mathrm{R}$ and binary operation $\circ$ is defined by (1.6) where $c \in \mathrm{R}_+ \setminus \{0, 1\}$.

Problem C: In particular, the authors of the present article would like to know the solution of Problem B in the case of $c = -1$, in more details, find the general solution of functional equation

$$f\left(x + y - \frac{xy}{x+y}\right) + g\left(\frac{xy}{x+y}\right) = h(x) + h(y) \qquad (x, y \in \mathrm{R}_+)$$

with unknown functions $f, g, h : \mathrm{R}_+ \to \mathrm{R}$.

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