A RELATIONSHIP BETWEEN GLOBAL OFFENSIVE ALLIANCE AND DOMINANT SETS

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Abstract
In this paper we show the relationship between global offensive alliance and dominant sets in the graphs and the random graphs. A global offensive alliance in a graph G=(V, E) is a subset S of V such that for every vertex v not in S at least half of the vertices in the closed neighbourhood of v are in S and a dominating set for a graph G is a subset D of its vertices, such that any vertex of G is either in D, or has a neighbour in D.

Keywords: *graph, global offensive alliance, dominant set*

1. Introduction

So, in this paper we will examine the relationship between global offensive alliance and dominant sets. By definition:

a global offensive alliance in a graph G=(V, E) is a subset S of V such that for every vertex v not in S at least half of the vertices in the closed neighbourhood of v are in S.

By definition:

a dominating set for a graph G is a subset D of its vertices, such that any vertex of G is either in D, or has a neighbour in D.

Nowadays, properties global offensive alliance in a graph and dominating set for a graph are intensively investigated. The literature on both properties is extensive and diverse. Balakrishnan and his co-authors explore the complexities of finding optimal global alliances (Balakrishnan et al., 2006). Blidia and his co-authors give a characterizations of trees with an unique minimum locating-dominating sets (Blidia et al., 2011). Bouzefrane and his co-authors was studied the global offensive alliance number of a tree (Bouzefrane et al., 2009). Bouzefrane give a comprehensive study for global alliances in trees (Bouzefrane et al., 2011). Bouzefrane and his co-authors was studied the global offensive alliance in unicycle graphs (Bouzefrane et al., 2018). Chellali give an interesting study on offensive alliances in bipartite graphs (Chellali, 2010). Chellali and Haynes investigated the trees with unique minimum paired domination sets (Chellali et al., 2004). Chellali and Haynes give a characterization of trees with unique minimum double domination sets (Chellali et al., 2010). Chellali and Rad investigated the trees with unique Roman dominating functions of minimum weight (Chellali et al., 2014). Chellali and Volkmann show the independence and global offensive alliance in graphs (Chellali et al., 2010). Cockayne, Gamble and Shepherd give an upper bound for the k-domination number of a graph (Cockayne et al., 1985). Favaron, Fricke, Goddard, S. M. Hedetniemi, S. T. Hedetniemi, Kristiansen, Laskar and Skaggs show the offensive alliances in graphs (Favaron et al., 2004). Fiedler investigated a property of eigenvectors of nonnegative symmetric matrices and its application to graph theory (Fiedler, 1975). Fischermann presented the block graphs with unique minimum dominating sets (Fischermann, 2001). Fischermann, Rautenbach and Volkmann show the maximum graphs with an unique minimum dominating set (Fischermann et al., 2003). Fischermann and Triesch give the domination parameters and their unique realizations (Fischermann, 2002). Fischermann and Volkmann give an unique minimum domination in trees (Fischermann et al., 2002). Fischermann and Volkmann show the cactus graphs with unique minimum dominating sets (Fischermann et al., 2003). Fischermann and Volkmann presented the unique independence, upper domination and upper irredundance in graphs (Fischermann et al., 2003). Fischermann, Volkmann and Zverovich show the unique irredundance, domination and independent domination in graphs (Fischermann et al.,2005). Harutyunyan show the some bounds on alliances in trees (Harutyunyan, 2010). Haynes, S. T. Hedetniemi and Henning investigate the global defensive alliances in graphs too (Haynes et al., 2003). Also from Haynes, S. T. Hedetniemi and Henning is the characterization of trees with equal domination and global strong alliance numbers (Haynes et al., 2004). Hedetniemi and Laskar show the connected domination in graphs (Hedetniemi et al., 1984). S. M. Hedetniemi, S. T. Hedetniemi and Kristiansen present a comprehensive study of alliances in graphs (Hedetniemi et al., 2004). Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi present a comprehensive study alliances in graphs (Kristiansen et al., 2004). Rodriguez-Velazquez and Sigarreta give a spectal study of alliances in graphs too ((Rodriguez-Velazquez et al., 2007). Rodriguez-Velazquez and Sigarreta show the offensive alliances in cubic graphs (Rodriguez-Velazquez et al., 2006a). Rodriguez-Velazquez and Sigarreta show the global offensive alliance number of a graph (Rodriguez-Velazquez et al., 2006b). Rodriguez-Velazquez and Sigarreta investigates the defensive alliances and line graphs (Rodriguez-Velazquez et al., 2006c). Rodriguez-Velazquez and Sigarreta show the global offensive alliance number of a graph (Rodriguez-Velazquez et al., 2006d). Túri investigated the transfer and storage of information with the help of dominating sets in graph, with particular regard to the organization of transport (Túri, 2022).

2. Examples and result

We give some examples of sets global offensive alliance in graphs.

First, we consider the simple (see Figure 1.). Here we denote with red colour the set of a global offensive alliance.

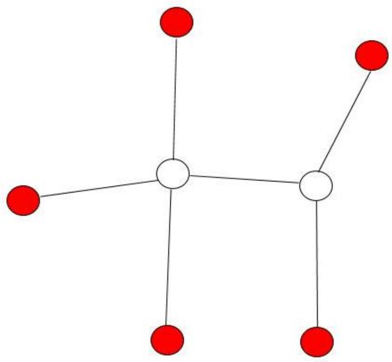

Figure 1.

Now we consider the same graph, but with a different set of global offensive alliance (see Figure 2.). Here we denote with red colour the set of a global offensive alliance.

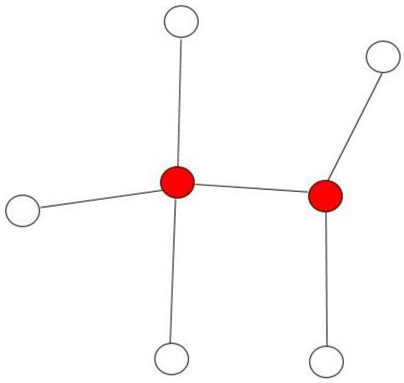

Figure 2.

Let consider an another graph. We show that there are several sets of global offensive alliance for this graph as well.

First, we consider the graph below (see Figure 3.). Here we denote with red colour the set of a global offensive alliance.

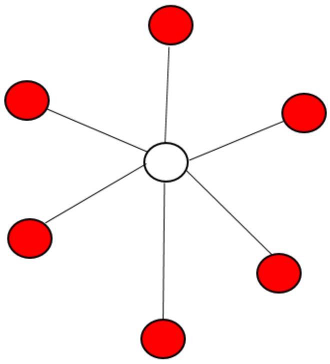

Figure 3.

First, consider the graph below (see Figure 4.). Here we denote with red colour the set of a global offensive alliance.

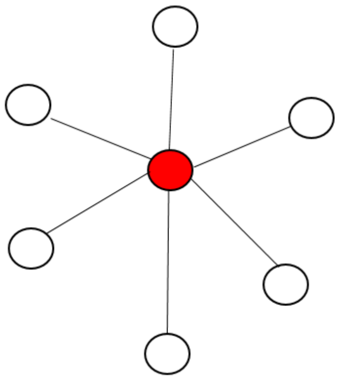

Figure 4.

If we look at the graphs above (Figure 1-4), we can see that the set of elements marked in red is also a dominant set. That is, it satisfies the definition of dominating set, i.e. the following: dominating set for a graph G is a subset D of its vertices, such that any vertex of G is either in D, or has a neighbour in D.

Thus, we can assume that the properties global offensive alliance and global offensive alliance are the same. However, this is not the case, as the next theorem says.

Theorem. *Properties global offensive alliance and properties dominating are not equivalent. Property* global offensive alliance *leads to property dominance. But the reverse is not true. If the dominance property exists, property global offensive alliance does not necessarily follow from it.*

Proof. Let's consider the two definitions again.

The property of global offensive alliance is defined as: a global offensive alliance in a graph G=(V, E) is a subset S of V such that for every vertex v not in S at least half of the vertices in the closed neighbourhood of v are in S.

The property of dominance is defined as: a dominating set for a graph G is a subset D of its vertices, such that any vertex of G is either in D, or has a neighbour in D.

We show that property of global offensive alliance follows from the property of dominance. This is also true, because if we consider vertices outside of S (we can choose S as the dominant set, i.e. D), each of them has a neighbour in S. Moreover, according to the definition, it has at least as many neighbours in S as outside it. So, the global offensive alliance property always results in the dominant property. However, the converse is not true: it results from the fact that every vertex in a non-dominant set has a neighbour in a dominant set, but not necessarily more, i.e. usually less than what is required in the definition of global offensive alliance.

Let's immediately consider two examples that property of dominance is fulfilled, but property of global offensive alliance property of dominance is not true.

Consider the graph below (Figure 5.). Here we can see that the vertices marked in red form a dominant set but do not fulfil the definition of global offensive alliance.

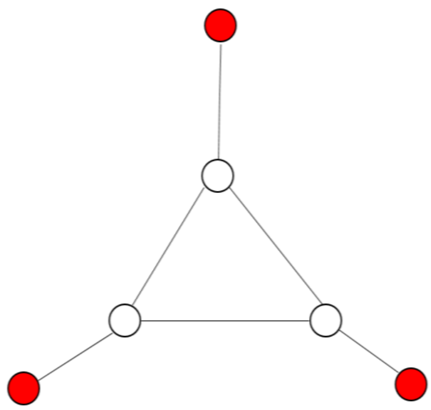

Figure 5.

Consider the graph (see Figure 6.). Similarly, to the previous one, the vertices marked in red form a dominating set, but the property of global offensive alliance in the definition is not fulfilled here either.

Thus, it does not satisfy the already known definition, i.e. it does not exist that the property of global offensive alliance is defined as: a global offensive alliance in a graph G=(V, E) is a subset S of V such that for every vertex v not in S at least half of the vertices in the closed neighbourhood of v are in S.

But of course the property of dominating set is defined as: a dominating set for a graph G is a subset D of its vertices, such that any vertex of G is either in D, or has a neighbour in D is valid.

We note that it would perhaps be worthwhile to introduce an intermediate definition as well, but discussing this does not fit into the scope of our investigations.

In the possible new definition, instead of halving, it could also be, say, a "k" variable.

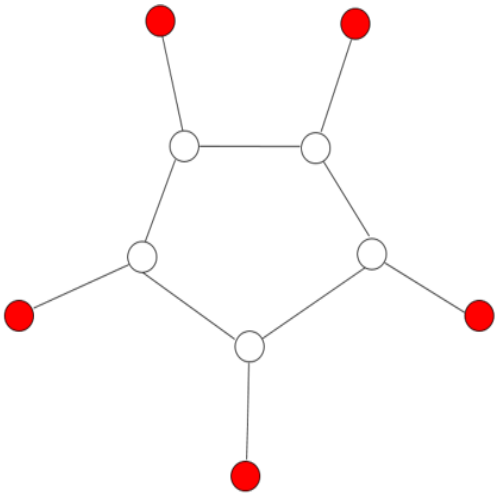

Figure 6.

So, the proof is complete.

Remark. If we also study global offensive alliances of random graphs. In particular, it is proved that if p(log n) 1/2 → ∞ then with high probability G(n, p) has a global offensive alliance of size at most can if c > 1/2 and no global offensive alliance of size at most can if c < 1/2.

3. Application

In the work „The transfer and storage of information with the help of dominating sets in graph, with particular regard to the organization of transport" (see [32]), if the selected dominant set is replaced by the global offensive alliance trait, the above still works. It is true that in general it can be said that more resources are required, but since there are usually more connections, it is more secure. Thus, after the exchange, we actually get an over insured system.

The following table summarizes the advantages and disadvantages of the global offensive alliance and dominant methods in terms of the above application.

Table 1.

	global offensive alliance	**dominating**
advantage	usually over insured	not redundant
disadvantage	usually redundant	no multi-level insurance

4. Summary

In this paper, we show two graph-theoretic properties: the global offensive alliance and dominant properties. We illustrate these properties on several graphs. Both of the described graph theoretical properties play a significant role in various applications. We show that the two properties are very similar to each other: property global offensive alliance in a graph leads to property dominant properties, but we also show that the reverse is not true.

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