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GREEN'S FUNCTION TECHNIQUE FOR STEPPED BEAM VIBRATIONS

Abderrazek Messaoudi
PhD student, Institute of Applied Mechanics, University of Miskolc
3515 Miskolc-Egyetemváros, e-mail: abderrazek.messaoudi@uni-miskolc.hu

György Szeidl
Professor emeritus, Institute of Applied Mechanics, University of Miskolc
3515 Miskolc-Egyetemváros, e-mail: gyorgy.szeidl@uni-miskolc.hu

László Péter Kiss
Associate professor, Institute of Applied Mechanics, University of Miskolc
3515 Miskolc-Egyetemváros, e-mail: laszlo.kiss@uni-miskolc.hu

Abstract
The study investigates stepped heterogeneous beams fixed at two endpoints. The eigenfrequencies for both unloaded and axially loaded problems are derived through the transformation of classical eigenvalue problems into homogeneous Fredholm integral equations. These equations can numerically be solved by reducing them to algebraic eigenvalue problems. The study assesses the effects of step location, bending stiffness and, axial force on natural frequencies. Through a comprehensive exploration of these mathematical frameworks, the paper aims to contribute to the structural vibration analysis of stepped beams. The study's outcomes confirm the suitability of the presented method.

Keywords: *Green's function, stepped beam, axial force, Fredholm integral equation, eigenfrequency*

1. Introduction

The importance of tackling vibration challenges in stepped beams lies in their practical applications across various engineering scenarios. These beams serve as effective models for studying vibrations in structures like robot arms and tall buildings. A number of researchers have conducted in-depth studies on the vibration behavior of axially loaded stepped Euler–Bernoulli beams, with significant work presented by (Raju et al., 1994). Their article focuses on studying the free vibrations of stepped beams under partial stress induced by follower forces. To address the problem, the authors employed the Galerkin formulation along with the method introduced by Kikuchi. In a paper (Naguleswaran, 2004), the investigation delves into the free vibration behavior of a beam featuring a single step. Notably, this beam undergoes loading from an axial force, which alternates between compressive and tensile states at the step. Green's theorem and function were introduced nearly two centuries ago by Green (Green, 1828), in the course of solving an electrostatic problem. One key advantage of this approach is its flexibility, as it can model any load and boundary conditions while offering a simple and efficient solution. The method has been applicable to wide kinds of engineering problems – see e. g., papers (Stakgold et al., 2011) and (Hozhabrossadati et al., 2018). Green's function method may be applied to solve vibration issues with stepped beams. This method's benefit is most evident when dealing with stepped beams with a large number of uniform segments (Kuklam et al., 2007). The research of (De Rosa et al., 1995) is dedicated to the free vibrations of stepped beams under the assumption of the Euler–Bernoulli beam theory. These beams are positioned on an elastic foundation with varying stiffness at each step. The frequency equation for this system is solved numerically. Study (Su et al., 2024) investigated the dynamic behavior of a functionally graded material stepped beam under various boundary conditions. Utilizing the Euler–Bernoulli beam theory, both free and forced vibrations of the beam were analyzed using the transfer matrix method. In (Kiss et al., 2022) this method was used to investigate the effect of the axial load on the natural frequencies of heterogeneous fixed–fixed beams with intermediate roller support, but the analysis only concerns the vibration of uniform beams. The previous investigations are extended to stepped members (Kiss et al., 2024).

Drawing from a thorough review of open literature, this article investigates both the free and loaded vibration frequencies of nonhomogeneous stepped beams with two segments. The study's outcomes are solved numerically by reducing homogeneous Fredholm integral equations to algebraic eigenvalue problems. Exemplary numerical results are given to show the application of the presented approach.

2. MECHANICAL MODEL

The depicted heterogeneous fixed-fixed stepped beam is illustrated in *Figure 1*. Cross section segments A_1 and A_2 maintain uniformity along their respective lengths. The E-weighted center line, abbreviated as the center line, aligns with the $\hat{x}$-axis of the coordinate system $\hat{x}, \hat{y}, \hat{z}$. The origin is situated at the left end of this center line. It is assumed that the $\hat{x}\hat{z}$ plane serves as a symmetry plane for the beam. Additionally, the modulus of elasticity E is presumed to adhere to the relationship $E(\hat{y},\hat{z}) = E(-\hat{y},\hat{z})$ across cross-sectional regions A_1 and A_2. The beam's length is denoted as L, with the discontinuity in cross sections occurring at point $\hat{b}$.

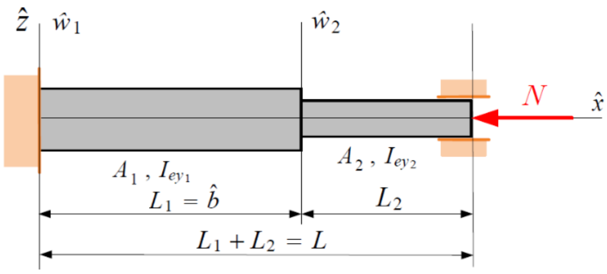

Figure 1. *Heterogeneous beam*

Equilibrium problems of non-uniform beams with cross-sectional heterogeneity are governed by the following ODEs (Baksa et al., 2009)

$$I_{ey_1}\frac{\mathrm{d}^4\hat{w}_1}{\mathrm{d}\hat{x}^4}=\hat{f}_{y1}(\hat{x}),\ \ \hat{x}\in[0,\hat{b}];\qquad I_{ey_2}\frac{\mathrm{d}^4\hat{w}_2}{\mathrm{d}\hat{x}^4}=\hat{f}_{y2}(\hat{x}),\ \ \hat{x}\in[\hat{b},L] \tag{1}$$

where $\hat{w}_i(\hat{x})$ $(i=1,2)$ is the vertical displacement of the material points on the center line, $\hat{f}_{zi}(\hat{x})$ is the intensity of the distributed load acting on the center line while I_{ey_i} is the E-weighted moment of inertia of the cross-section part. In the following, we switch to dimensionless variables – these are introduced by dividing the circumflexed notations with the L length. Moreover, the notation is also introduced for the derivatives: $\mathrm{d}^n\ldots/\mathrm{d}x^n=\ldots^{(n)}$. ODEs (1) are associated with the following boundary and continuity conditions of *Table 1*.

Table 1. *Boundary and continuity conditions*

Boundary conditions	
$w_1(0)=0\,, w_1^{(1)}(0)=0\,, w_2(\ell)=0\,, w_2^{(1)}(\ell)=0$	
Continuity conditions	
$w_1(b-0)=w_2(b+0)$	$w_1^{(1)}(b-0)=w_2^{(1)}(b+0)$
$I_{ey_1}w_1^{(2)}(b-0)=I_{ey_2}w_2^{(2)}(b+0)$	$I_{ey_1}w_1^{(3)}(b-0)=I_{ey_2}w_2^{(3)}(b+0)$

In case the Green function $G(x,\xi)$ for the boundary value problem determined by ODEs (1) and the boundary and continuity conditions presented in *Table 1* is available, the solution for the dimensionless deflection $w(x)$ is given by the integral .

$$w(x)=\int_0^\ell G(x,\xi)f(\xi)\mathrm{d}\xi. \tag{2}$$

3. Vibration of beams

3.1. Free vibration

When focusing on the vibrations, let us consider the forthcoming:

$$f(\xi)=\begin{cases}\frac{\rho_{a1}A_1L^4}{I_{ey_1}}\omega^2 w(\xi) & \text{if}\quad \xi\in[0,b],\\ \frac{\rho_{a2}A_2L^4}{I_{ey_2}}\omega^2 w(\xi) & \text{if}\quad \xi\in[b,\ell].\end{cases}=\underbrace{\frac{\rho_{a1}A_1L^4}{I_{ey_1}}\omega^2}_{\lambda}\,w(\xi)\begin{cases}1 & \text{if}\quad \xi\in[0,b],\\ \frac{\rho_{a2}A_2I_{ey_1}}{\rho_{a1}A_1I_{ey_2}} & \text{if}\quad \xi\in[b,\ell].\end{cases} \tag{3}$$

In this context, ρ_{ai} represents the average density in the area A_i while ω symbolizes the circular frequency of the vibration. Recalling (2), the eigenvalue problem for λ is governed by the homogeneous Fredholm integral equation

$$w(x)=\lambda\int_0^\ell G(x,\xi)w(\xi)\begin{Bmatrix}1 & \text{if}\quad \xi\in[0,b]\\ \frac{\rho_{a2}A_2I_{ey_1}}{\rho_{a1}A_1I_{ey_2}} & \text{if}\quad \xi\in[b,\ell]\end{Bmatrix}\mathrm{d}\xi. \tag{4}$$

The Green function of the boundary value problem is generally sought in four intervals as detailed in (Kiss et al., 2024).

$$G(x,\xi)=\begin{cases}G_{11}(x,\xi) & \text{if} \quad x,\xi \in [0,b],\\ G_{21}(x,\xi) & \text{if} \quad x \in [b,\ell] \text{and} \xi \in [0,b],\\ G_{12}(x,\xi) & \text{if} \quad x \in [0,b] \text{and} \xi \in [b,\ell],\\ G_{22}(x,\xi) & \text{if} \quad x,\xi \in [b,\ell].\end{cases} \tag{5}$$

$$\begin{aligned}G_{11}(x,\xi) = {} & (-(\xi^3/(12\,I_{\mathrm{e\gamma1}})) \pm (\xi^3/(12\,I_{\mathrm{e\gamma1}}))) + ((3/(12\,I_{\mathrm{e\gamma1}}))\,\xi^2 \\ & \pm -(3/(12\,I_{\mathrm{e\gamma1}}))\,\xi^2)\cdot x + ((3\xi/(12\,D\,I_{\mathrm{e\gamma1}}))\,[(\ell - b)^4 + \alpha(-4b^3\alpha\xi \\ & + 2\alpha b^2\xi^2 + 4b^3\xi - 2b^2\xi^2 + 2\ell^2\xi^2 - 4\ell^3\xi + 4b\ell^3 - 6b^2\ell^2 + 4\ell b^3 + \alpha b^4 \\ & - 2b^4)] \pm (3\xi/12))\cdot x^2 + (-1/(12\,D\,I_{\mathrm{e\gamma1}}))\,[(\ell - b)^4 + \alpha(4b\ell^3 \\ & - 6b^2\ell^2 + 4\ell b^3 + \alpha b^4 - 2b^4 - 6\ell^2\xi^2 + 4\alpha b\xi^3 - 6\alpha b^2\xi^2 + 4\ell\xi^3 - 4b\xi^3 \\ & + 6b^2\xi^2)] \pm -(1/12)\cdot x^3\end{aligned} \tag{6}$$

$$\begin{aligned}G_{21}(x,\xi) = {} & (1/(6\,D\,I_{\mathrm{e\gamma1}}))\cdot\xi^2\cdot(\ell - x)^2\cdot[3\ell^2x - 6\ell b^2 + 6b^3 - 3b^2x - 3b^2\xi + 6b^2\alpha\ell \\ & - 6b^3\alpha + 3b^2\alpha x - \xi\ell^2 + 4\xi\ell b - 2\xi\ell x + 2bx\xi - 4\xi\alpha b\ell + 3\xi\alpha b^2 \\ & - 2\xi b\alpha x]\end{aligned} \tag{7}$$

$$D = (\ell - b)^4 + \alpha b(\alpha b^3 - 2b^3 + 4\ell b^2 - 6\ell^2 b + 4\ell^3), \qquad \alpha = \frac{I_{ey2}}{I_{ey1}}, \tag{8}$$

$$\begin{aligned}G_{12}(x,\xi) = {} & (1/(6\,D\,I_{\mathrm{e\gamma1}}))\cdot x^2\cdot(\ell - \xi)^2\cdot[3\ell^2\xi - 6b^2\ell + 6b^3 - 3b^2\xi + 6b^2\alpha\ell \\ & - 6b^3\alpha + 3b^2\alpha\xi - x\ell^2 + 4x\ell b - 2x\ell\xi - 3xb^2 + 2xb\xi - 4x\alpha b\ell + 3x\alpha b^2 \\ & - 2xb\alpha\xi]\end{aligned} \tag{9}$$

$$
\begin{aligned}
G_{22}(x,\xi) = {} & (1\,/\,(12\,D\,I_{\mathrm{e}\gamma 2}))\,[-(\ell - b)(-2b\ell + \xi\ell + b\xi)(-2\ell^2 b^2 + 2b\ell^2\xi + \ell^2\xi^2 \\
& + 2b^2\ell\xi - 4b\ell\xi^2 + b^2\xi^2) - b\alpha(6b\xi\ell^4 - 12\ell^2\xi b^3 + 6b^3\xi\ell^2\alpha + 12\ell^2 b^2\xi^2 \\
& + 4\ell^3\xi^3 - 12b\ell^3\xi^2 - 4\ell^4 b^2 + 8\ell^3 b^3 - 4b^3\ell^3\alpha + 2\xi^3 b^3 - b^3\xi^3\alpha \\
& - 4b^2\xi^3\ell)] \pm (\xi^3\,/\,(12\,I_{\mathrm{e}\gamma 2})) - (3\,/\,(12\,D\,I_{\mathrm{e}\gamma 2}))\,[(\ell - b)(4\ell^3 b\xi - 2\ell^3 b^2 \\
& - \ell^3\xi^2 - 5\ell^2\xi^2 b + 4\ell^2 b^2\xi - 2\ell^2 b^3 + 4\ell b\xi^3 - 5\ell\xi^2 b^2 + 4\ell b^3\xi - \xi^2 b^3) \\
& - b\alpha(-b^3\alpha\xi^2 - 2\ell^4 b - 8\ell b^3\xi + 4b^3\alpha\ell\xi - 4\ell b\xi^3 + 4\ell^4\xi + 4\ell^2\xi^3 - 4\ell^3\xi^2 \\
& + 4\ell\xi^2 b^2 + 2\xi^2 b^3 + 4\ell^2 b^3 - 2b^3\ell^2\alpha)] \pm (-3\xi^2\,/\,(12\,I_{\mathrm{e}\gamma 2}))\cdot x \\
& + (3\,/\,(12\,D\,I_{\mathrm{e}\gamma 2}))\,[(\ell - b)(-4b^2\xi^2 - 4b^2\ell^2 - 4\xi^2\ell^2 + 2b\xi^3 + b^3\xi + \xi\ell^3 \\
& + 2\xi^3\ell + 5b\xi\ell^2 - 4b\xi^2\ell + 5b^2\xi\ell) - b\alpha(4b^2\xi^2 + 4b^2\ell^2 - 2b\xi^3 - 2b^3\xi \\
& - 4b\ell^3 + 4\xi\ell^3 + b^3\alpha\xi - 4b^2\xi\ell)] \pm (3\xi\,/\,(12\,I_{\mathrm{e}\gamma 2}))\cdot x^2 \\
& - (1\,/\,(12\,D\,I_{\mathrm{e}\gamma 2}))\,[(\ell - b)(b - 2\xi + \ell)(2b\xi - 4b\ell - 2\xi^2 + \ell^2 + 2\xi\ell \\
& + b^2) + b\alpha(4\xi^3 + 4\ell^3 - b^3\alpha - 6b\xi^2 - 4b^2\ell - 12\xi\ell^2 + 2b^3 + 12b\xi\ell)] \\
& \pm (-1\,/\,(12\,I_{\mathrm{e}\gamma 2}))\cdot x^3
\end{aligned}
\tag{10}
$$

While the dimensionless Green function is given by

$$
\mathcal{G}(x,\xi) = \begin{cases}
\mathcal{G}_{11}(x,\xi) = I_{ey_1} G_{11}(x,\xi) & \text{if} \quad x,\xi \in [0,b], \\
\mathcal{G}_{21}(x,\xi) = I_{ey_1} G_{21}(x,\xi) & \text{if} \quad x \in [b,\ell] \text{ and } \xi \in [0,b], \\
\mathcal{G}_{12}(x,\xi) = I_{ey_2} G_{12}(x,\xi) & \text{if} \quad x \in [0,b] \text{ and } \xi \in [b,\ell], \\
\mathcal{G}_{22}(x,\xi) = I_{ey_2} G_{22}(x,\xi) & \text{if} \quad x,\xi \in [b,\ell],
\end{cases}
\tag{11}
$$

thus,

$$
w(x) = \chi \int_0^{\ell} \mathcal{G}(x,\xi) w(\xi) \begin{Bmatrix} 1 & \text{if} \quad \xi \in [0,b], \\ \kappa & \text{if} \quad \xi \in [b,\ell]. \end{Bmatrix} \mathrm{d}\xi
\tag{12}
$$

with the new eigenvalue being

$$
\chi = \frac{\lambda}{I_{ey_1}} = \frac{\rho_{a1} A_1 L^4}{I_{ey_1}} \omega^2, \qquad \text{and} \qquad \kappa = \frac{\rho_{a2} A_2 I_{ey_1}}{\rho_{a1} A_1 I_{ey_2}}.
\tag{13}
$$

3.2. Example 1

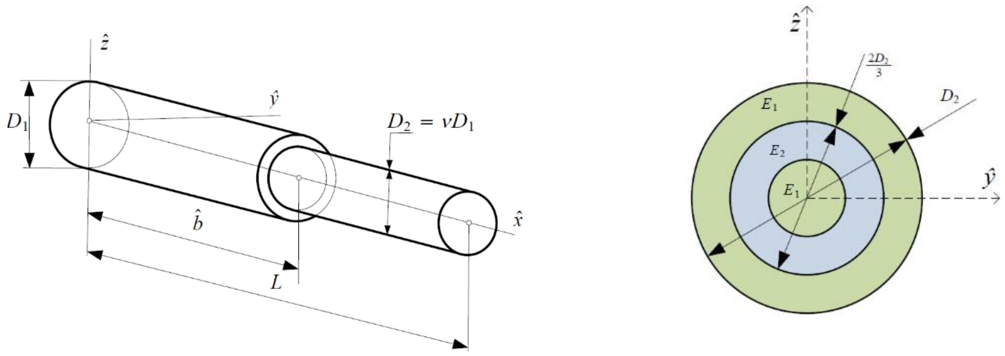

Figure 2. *Stepped beam with circular cross section*

Consider the stepped beam shown in *Figure 2*. We assume that $[\nu = 1$ if $\hat{x} \in 0, \hat{b})]$ $(\nu = 0.95, 0.90, 0.85, 0.80, 0.75$ if $\hat{x} \in (\hat{b}, L])$. It is also assumed that $D_1 = 50$ mm, $D_2 = \nu D_1$, $E_1 = E_{aluminum} = 0.71 \times 10^5$ N/mm^2 while $E_2 = E_{steel} = 2.0 \times 10^5$ N/mm^2. The length L of the beam is 4000 mm, the location of the step is given by the parameter $\hat{b}$. The surface densities have the following values: $\rho_1 = \rho_{\text{aluminium}} = 2710$ kg/10^{9}mm^3, $\rho_2 = \rho_{\text{steel}} = 7850$ kg/10^{9}mm^3.

Given the preceding circumstances, *Table 2* presents the distinctive data pertaining to the diverse cross-sectional areas. The eigenvalue problem presented in *equations (12)* and *(13)* has been effectively addressed through numerical solutions employing a solution algorithm based on the boundary element method . This method is detailed in, e. g., (Szeidl et al., 2020).

Figure 3 shows the computational results for $\sqrt{\chi_1}/4.73004^2$ in the function of the dimensionless parameter b. Each curve in this figure corresponds to a different value of the parameter α. With λ_1 it follows form (19) that

$$\omega_1 = \frac{1}{L^2}\sqrt{\frac{I_{ey_1}}{\rho_{a1}A_1}\chi_1}\,. \tag{14}$$

Table 2. *Data for the cross sections*

ν	$\rho_{a1} = \rho_{a2}$	I_{ey1}	I_{ey2}	$\alpha = \frac{I_{ey2}}{I_{ey1}}$	κ
	kg/mm^3	kg mm^3/s^2			
0.95			0.237115×10^{14}	0.814	1. 108033581
0.90			$0.1\ 91000 \times 10^{14}$	0.65609	1. 234586718
0.85	4.423333×10^{-6}	0.291115×10^{14}	0.1519640×10^{14}	0.522	1. 384099617
0.80			0.1192408×10^{14}	0.409	1. 564792176
0.75			0.0921107×10^{14}	0.316	1. 780063291

Assume that $\nu = 0.8$ and $b = 0.5$. Then

$$\sqrt{\chi_1} = 4.73004^2 \times 0.87628529 = 19.60537475$$

and

$$\omega_1 = \frac{1}{4000^2} \times \left(\sqrt{\frac{0.291115 \times 10^{14}}{4.423333 \times 10^{-6} \times \pi(((50)/2))^2}} \right) \times 19.60537475 = 70.94106344 \ \mathrm{r/s}.$$

If the beam has no steps and is made of homogeneous steel.

$$\sqrt{\chi_1} = 4.73004^2 \times 1 = 22.373\,278$$

and

$$\omega_1 = \frac{1}{4000^2} \times \left(\sqrt{\frac{6.135923152 \times 10^{13}}{7.85 \times 10^{-6} \times \pi(25)^2}} \right) \times 22.373\,278 = 88.226\,65 \ \mathrm{r/s}.$$

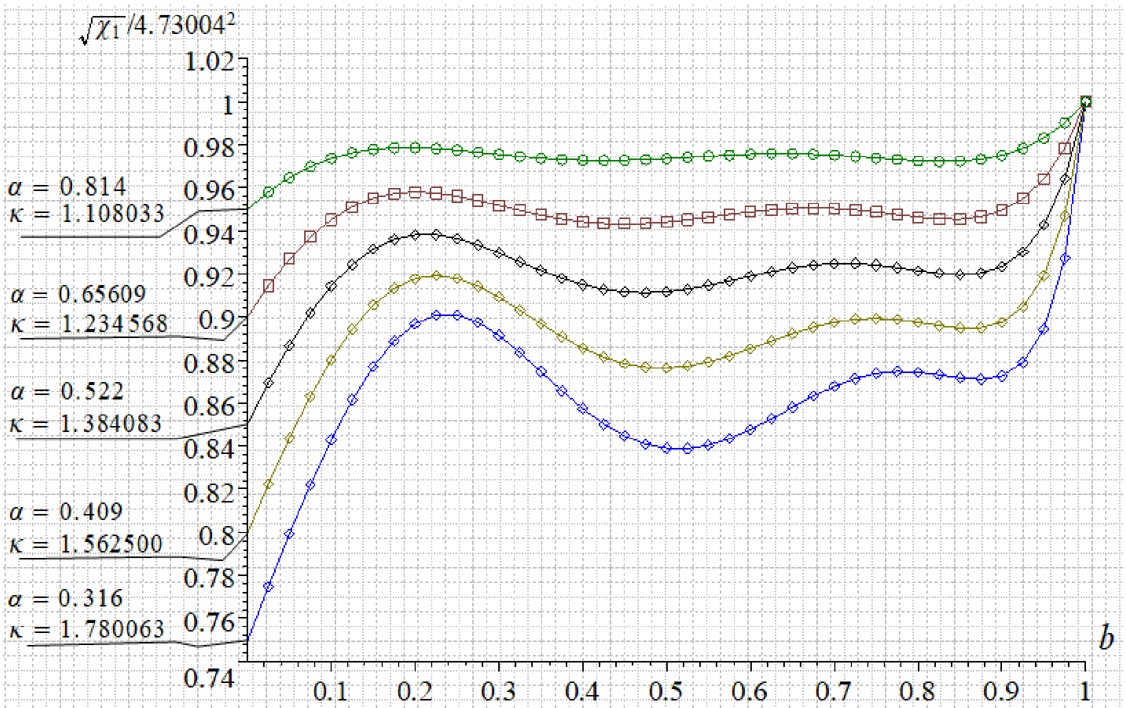

Figure 3. *The first eigenvalue as a function of b; α and κ are parameters*

These results were compared with finite element calculations conducted using Ansys. For mesh generation, a total of 492 uniform hexahedral elements (SOLID185) were utilized to discretize the geometry. A strong agreement has been observed as it is listed in *Table 3*.

Table 3. *Comparison to FEM results*

	New model solution _eigenfrequency [Hz]	Ansys solution _eigenfrequency [Hz]	Relative error %
Stepped beam	$\dfrac{88.226\,65}{2\pi} = 14.04$	13.98	0.42%
Uniform beam	$\dfrac{70.941063}{2\pi} = 11.29$	11.24	0.44%

When computing the relative error, our solution served as the denominator.

3.3. Axially loaded vibration

Our goal now is to elucidate how the axial load influences the natural frequencies of the stepped beam. Utilizing the dimensionless Green functions, we analyze the eigenvalue problems that arise, which are governed by the homogeneous Fredholm integral equations.

- In the case of a compressive force:

$$w(x) = \chi \int_0^\ell \mathcal{G}_c(x,\xi) w(\xi) \begin{Bmatrix} 1 & \text{if} & \xi \in [0,b], \\ \kappa & \text{if} & \xi \in b,\ell]. \end{Bmatrix} \mathrm{d}\xi\,, \tag{15}$$

and

- In the case of a tensile force:

$$w(x) = \chi \int_0^\ell \mathcal{G}_t(x,\xi) w(\xi) \begin{Bmatrix} 1 & \text{if} & \xi \in [0,b], \\ \kappa & \text{if} & \xi \in [b,\ell]. \end{Bmatrix} \mathrm{d}\xi. \tag{16}$$

Here, the eigenvalue sought, denoted by χ, and κ are determined by *equation (13)*. Consequently, we will pursue numerical solutions for the aforementioned problems, utilizing the data pertinent to the stepped beams discussed in Subsection 3.2. For a detailed explanation, we refer to papers (Kiss et al,. 2022) and (Kiss et al., 2024) for a comprehensive understanding of the principles employed in calculating the Green functions $\mathcal{G}_c(x,\xi)$ and $\mathcal{G}_t(x,\xi)$.

3.4. Example 2

This time, we shall assume $\nu = 0.90$; then $\alpha = 0.65609$ and $\kappa = 1.234586718$. These data are taken from *Table 2.* The first eigenfrequency and the dimensionless critical force can be calculated by utilizing the data presented in *Tables 4* and *5*, which contain some further data that are also utilized in the computations.

Table 4. *Values of* χ_1

	$\sqrt{\chi_1(b)}/4.73004^2$				
ν	$b=0.2$	$b=0.4$	$b=0.5$	$b=0.6$	$b=0.8$
0.9	0.95804888	0.94421338	0.94412264	0.94874218	0.94639241

In the following, we will need the value of the smallest critical force for the mentioned stepped beams that can result in buckling. The equilibrium problems of axially loaded beams with cross-sectional heterogeneity are governed by the differential equations.

$$\begin{aligned} H_{1a}(w_1(x)) &= I_{ey_1} w_1^{(4)} \pm N_1 L^2 w_1^{(2)} = f_{z1}(x), \;\; x \in [0, b]; \\ H_{2a}(w_2(x)) &= I_{ey_2} w_2^{(4)} \pm N_2 L^2 w_2^{(2)} = f_{z2}(\hat{x}), \;\; x \in [b, l] \end{aligned} \tag{17}$$

where N_1 and N_2 ($N_1 > 0$, $N_2 > 0$) are the axial forces in the beam. The sign is (positive) [negative] if the considered axial force is (compressive) [tensile]. Differential *equations (17)* are associated with boundary and continuity conditions are presented in *Table 1.*

Table 5. *Lowest dimensionless critical (buckling) forces*

	$\sqrt{N_{2\,\mathrm{crit}}(b)}$				
ν	$b=0.2$	$b=0.4$	$b=0.5$	$b=0.6$	$b=0.8$
0.9	6.62368062	6.68734516	6.87217581	7.16006840	7.38138225

Let us denote the first eigenfrequency for [compression] {tension} by $[\omega_{1c}]$ $\{\omega_{1t}\}$. The first eigenfrequency of the unloaded beam is ω_1.

- *Results if* $b = 0.5$:

Table 6 contains the computed results in this case. It can be observed that the relationships are almost linear, but can better be approximated more accurately by the polynomials

$$\begin{aligned} \frac{\omega_{1c}^2}{\omega_{1\mathrm{noload}}^2} &= -4.389502 \times 10^{-2} \frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} - 0.954998 \frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 0.999483\,, \\ \frac{\omega_{1t}^2}{\omega_{1\mathrm{noload}}^2} &= -2.116397 \times 10^{-2} \frac{\mathcal{N}_2^2}{\mathcal{N}_{2crit}^2} + 0.958711 \frac{\mathcal{N}_2}{\mathcal{N}_{2crit}} + 1.000204 \end{aligned} \tag{18}$$

Table 6. *Eigenfrequency ratios as functions of the axial load*

$b = 0.5$		
$N/N_{crit} = \mathcal{N}_2/\mathcal{N}_{2crit}$	$\omega_{1c}^2/\omega_{1\text{noload}}^2$	$\omega_{1t}^2/\omega_{1\text{noload}}^2$
0.000	0.99996945	1.00003053
0.200	0.80647344	1.19119751
0.300	0.70872464	1.28601839
0.400	0.61024815	1.38036431
0.500	0.51098255	1.47426423
0.600	0.41085828	1.56774452
0.800	0.20770507	1.75354039
1.000	0.00000000	1.93792132

If b = 0 or b = 1 the beam behaves as if it were a fixed-fixed beam (there is no step in the beam). Then the quadratic approximation for $\omega_{1c}^2/\omega_{1\text{noload}}^2$ coincide with the quadratic approximation published in (Szeidl et al., 2005).

Table 7 presents a comparison of our model with results published in the literature for both free vibration and loaded vibration, confirming the accuracy of the model. The results demonstrate good agreement.

Table 7. *Comparison with literature results*

$\sqrt{\chi}$	Our results	Literature results
	$\alpha = 5, \quad \kappa = 1/\sqrt{5}, \quad b = 0.5$	
Free vibration	25. 959	25.959 [13]
	$\alpha = 0.4096, \ \kappa = 1.5625, \nu = 0.8, \ \mathcal{N}_2 = 10, b = 0.375$	
Loaded vibration (tensile force)	25. 700	25. 207 [2]

4. CONCLUSIONS

A definition is given for the Green functions of some coupled boundary value problems. The definition is a constructive one since it makes possible to calculate the Green function themselves. The eigenvalue problem for the free vibrations of the stepped beam was replaced by an eigenvalue problem governed by a homogeneous Fredholm integral equation. The classical vibration problem of the axially loaded stepped beam was also replaced by the Fredholm integral equation. These eigenvalue problems were solved numerically to reveal the effect of various parameters on the vibrational frequencies. It is a well known result that the equations

$$\frac{W^2{}_1}{W^2{}_{1\,no\,load}} = 1 \pm \frac{N^2{}_2}{N^2{}_{2\,critical}} \tag{19}$$

are the solutions to a similar problem for simply supported homogeneous and heterogeneous beams – in the later case cross sectional heterogeneity is assumed. According to the computational results *equation (19)* provide a good approximation for the fixed-fixed stepped beams as well. See *equation (18)* for a comparison. Finite element verifications were also made as control calculations.

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