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STABILITY AND ACCURACY ANALYSIS OF CLASSICAL TIME-STEPPING ALGORITHMS IN A UNIFIED CHARACTERISTIC-POLYNOMIAL FRAMEWORK

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Abstract

This paper presents a unified spectral framework for evaluating the stability and accuracy of classical time-stepping algorithms via the characteristic polynomials of their approximation operators. The Central Difference Method and the Wilson–θ scheme are analysed using the undamped single-degree-of-freedom oscillator, enabling numerical effects to be distinguished from physical behaviour. Stability is established through the Jury criterion, while numerical damping and phase error are quantified from the dominant eigenvalues. The Central Difference Method is conditionally stable and amplitude-preserving but exhibits a second-order negative phase error, whereas the Wilson–θ Method achieves unconditional stability at the cost of fourth-order numerical dissipation and a positive second-order phase error. The proposed framework provides a transparent basis for comparing integration schemes and highlights the trade-off between unconditional stability and long-time accuracy.

Keywords: *time-stepping algorithms, stability analysis, amplitude decay, period elongation, spectral analysis*

1. Introduction

The traditional approach to the numerical solution of dynamic initial-boundary value problems is based on the separate discretization of time and space variables. Within this framework, the spatial approximation, for example using the finite element method, results in a semi-discrete system of equations. Numerous time-stepping algorithms have been developed for the numerical solution of the resulting matrix differential equations describing temporal processes, in which the displacements are computed from initial values at discrete time instants that are close to each other.

These methods can be broadly classified into two groups. In explicit methods, the displacements at a given instant are determined from the dynamic equilibrium equations evaluated at the previous instant. In contrast, implicit procedures require that the solution at a given instant satisfy the dynamic equilibrium equation at that same instant. Explicit algorithms are computationally simpler; however, their stability is strongly influenced by the size of the time step. By contrast, implicit methods can achieve stability that is largely independent of the time-step size.

Representative time-stepping methods include the central difference method, the Newmark method, its generalization the Hilbert–Hughes–Taylor (HHT) scheme, the Wilson–θ method, and the Bathe method (Györgyi, 2006). In recent years, the development of time-stepping algorithms has increasingly focused on improving stability, accuracy, and computational efficiency. High-order and parallel-in-time methods have been proposed to enhance the performance of numerical simulations of dynamic systems (Schütz et al., 2022). Furthermore, implicit–explicit formulations have gained considerable attention because they provide a favourable balance between numerical stability and computational cost when solving time-dependent differential equations (Arun et al., 2021; Hu & Shu, 2021).

Despite their widespread use, classical time-stepping algorithms exhibit several well-known limitations, and the continuous development of modern time-stepping schemes indicates that stability, accuracy, and numerical dissipation remain active research topics. First, stability is achieved only under specific choices of numerical control parameters, and even within the stability domain the solution may suffer from slow convergence and reduced accuracy due to numerical damping (AD) and numerical dispersion (PE) (Kwon et al., 2020). Second, these methods typically require both the initial displacements and their first derivatives to be prescribed (Lee et al., 2024), but the initial condition can be given by six different ways (Tóth, 2016; Tóth, 2018). Third, they provide a stable numerical solution primarily for displacements as the fundamental variable (Tóth, 2016).

This study provides a critical spectral analysis of classical time-stepping algorithms whose limitations have motivated the development of time-integrating hp-FEM approaches. Specifically, the Central Difference Method and the Wilson–θ Method are examined using the undamped harmonic oscillator as a benchmark problem, allowing numerical effects to be clearly separated from physical behaviour.

The analysis focuses on stability, amplitude decay (AD), and period elongation (PE). The stability and accuracy properties of these schemes are investigated within a unified framework based on the characteristic polynomials of the approximation operators. Stability is established using the Jury criterion, while amplitude decay and period elongation are quantified through the modulus and complex arguments of the dominant eigenvalues.

Although the present study focuses on classical schemes, their analysis remains essential for understanding the behaviour and limitations of modern time-stepping and time-integration methods.

2. Model and Definitions

The time-stepping algorithms will be examined using the example of a material point performing undamped free vibration with one degree of freedom, the mechanical model of which is shown in *Figure 1* m [kg] is the mass of the material point, k [N/m] is the spring constant. The following expression establishes a relationship between these two characteristics:

$$\omega = \sqrt{\frac{k}{m}}, \tag{1}$$

where ω is the natural circular frequency of the system $(0 < k, m, \omega \in \mathbb{R})$.

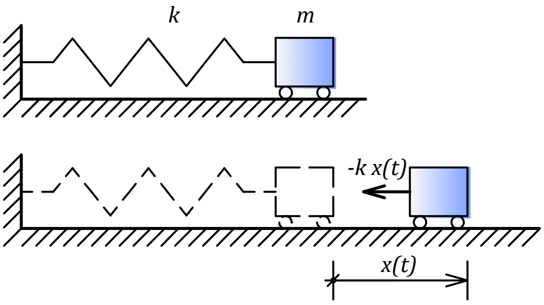

***Figure 1**. Single-degree-of-freedom spring–mass system. The displacement $x(t)$ induces a restoring force of $-k\ x(t)$ in the opposite direction.*

By applying Newton's second law and (1), we obtain the equation of motion for the system:

$$\ddot{x} + \omega^2 x = 0, \tag{2}$$

in which $x = x(t)$ is the displacement function ($x: \mathbb{R} \to \mathbb{R}$). If the initial conditions are given in the following form: $x_0 = x(0)$ and $v_0 = \dot{x}(0)$, the displacement function is given by the following analytical solution (Szeidl et al., 2020):

$$x(t) = x_0 \cos \omega\, t + \frac{v_0}{\omega} \sin \omega\, t. \tag{3}$$

The analytical solution of an undamped harmonic oscillator is a vibration with constant amplitude and period. Thus, any amplitude decay or period elongation arises solely from the characteristics of the numerical algorithm. Since the normal modes of semi-discrete finite element systems follow the same differential equation, the model is an ideal tool for investigating numerical damping (AD) and phase error (PE).

Applying time-stepping algorithms to the motion equilibrium of a material point performing undamped free vibration with one degree of freedom, we obtain the following recursive matrix equation (Györgyi, 2006; Bathe et al., 1976):

$$\underline{\mathbf{z}}_{n+1} = \underline{\underline{\mathbf{A}}}\, \underline{\mathbf{z}}_n, \tag{4}$$

where the state variables are stored in the vector $\underline{\mathbf{z}}_n$, and $\underline{\mathbf{z}}_{n+1}$ if the $t = n\,\Delta t$, or $t = (n+1)\,\Delta t$, and matrix $\underline{\underline{\mathbf{A}}}$ is the approximation operator ($n \in \mathbb{N}$). The Δt [s] is the magnitude of the time step ($0 < \Delta t \in \mathbb{R}$). From (4) we can get:

$$\underline{\mathbf{z}}_n = \underline{\underline{\mathbf{A}}}^n\, \underline{\mathbf{z}}_0. \tag{5}$$

In terms of the stability of integration methods, there are procedures that are unconditionally stable and others that are only conditionally stable. An integration method is unconditionally stable if, for any initial conditions, the solution does not grow to infinity for any time step Δt, specially if the ratio of the time step and the period time is large ($\Delta t/T$). The method is only conditionally stable if the above statement is true only when $\Delta t/T$ is less than or equal to a certain value, which is generally referred to as the stability limit.

The stability criterion of the time-stepping schemes (Györgyi, 2006; Bathe et al., 1976):

- if the eigenvalues of the approximation operator are different, the maximum absolute value of the eigenvalues cannot be bigger than 1.
- if there are multiple eigenvalues, their absolute value must be less than 1.

The stability of the procedure does not guarantee the accuracy of the solution. Two fundamental errors occur during the numerical solution of the equation of motion: amplitude decay (numerical dissipation) and period elongation (numerical dispersion) (Györgyi, 2006; Bathe et al., 1972; Bathe et al., 1976).

For oscillatory problems the dominant eigenvalues (denoted by λ) of the approximation operator form a complex conjugate pair ($G, \varphi \in \mathbb{R}$):

$$\lambda = |G| e^{i\,\varphi}. \tag{6}$$

Due to the stability criterion, $|G| \leq 1$. If $|G| = 1$, the amplitude is preserved, but when $|G| < 1$, the phenomenon of numerical dissipation can be observed. This decay is purely numerical: the exact solution of the undamped oscillator has constant amplitude therefore, any loss of energy is caused only by the time-integration scheme (Bathe et al., 1972; Hilber et al., 1977; Depouhon et al., 2014). For this reason, recent time-integration research has placed increasing emphasis on the energy behaviour of numerical schemes, since excessive algorithmic dissipation may lead to nonphysical damping of the structural response (Kim et al., 2023).

The numerical circular frequency is defined by the following expression:

$$\omega_{num} = \frac{\varphi}{\Delta t}. \tag{7}$$

The associated numerical period time:

$$T_{num} = \frac{2\pi}{\omega_{num}} = \frac{2\pi\,\Delta t}{\varphi}. \tag{8}$$

If $\omega_{num} \neq \omega$ the discrete solution oscillates either slower or faster than the exact one. The resulting error is called numerical dispersion or phase error, and it is commonly expressed in relative form as:

$$p_{error} = \frac{T_{num} - T}{T} = \frac{\omega\,\Delta t}{\varphi} - 1 = \frac{r}{\varphi} - 1, \tag{9}$$

where $r = \omega\,\Delta t$ ($0 < r \in \mathbb{R}$). A positive value indicates period elongation (the numerical solution lags behind the exact motion), whereas a negative value corresponds to period shortening (the numerical solution leads the exact one) (Bathe et al., 1972; Hilber et al., 1977; Depouhon et al., 2014).

Note: Since the eigenvalues form a complex-conjugate pair $e^{\pm i\,\varphi}$, only the root located on the upper half of the unit circle is used to define the numerical frequency. The conjugate root carries no additional physical information.

3. The Central Difference Method (CDM)

When applying CDM, we use the following approximation in *equation (2)* for the second time derivative of the displacement function:

$$\ddot{x}_n = \frac{1}{\Delta t^2}\,(x_{n+1} - 2x_n + x_{n-1}), \tag{10}$$

in (10) x_n is the displacement at the time instant $t_n = n\,\Delta t$. Substituting expression (10) into *equation (2)* based on (4), we can write the following:

$$\underline{\mathbf{z}}_{n+1}^{CDM} = \underline{\underline{\mathbf{A}}}^{CDM}\,\underline{\mathbf{z}}_{n}^{CDM}. \tag{11}$$

In expression (11) $\underline{\mathbf{z}}_{n+1}^{CDM} = [x_{n+1};\ x_{n+1}\,]^T$, $\underline{\mathbf{z}}_{n}^{CDM} = [x_n;\ x_{n-1}\,]^T$ and $\underline{\underline{\mathbf{A}}}^{CDM} = \begin{bmatrix} 2 - \omega^2\,\Delta t^2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 - r^2 & -1 \\ 1 & 0 \end{bmatrix}$. The characteristic polynomial of the approximation operator $\underline{\underline{\mathbf{A}}}^{CDM}$ (whose roots are the eigenvalues):

$$p_{CDM}(\mu) = \mu^2 + (r^2 - 2)\,\mu + 1. \tag{12}$$

The roots of (12):

$$\lambda_{1;2} = 1 - \frac{r^2}{2} \pm \sqrt{\left(\frac{r^2}{2} - 1\right)^2 - 1}. \tag{13}$$

3.1. The Stability Of CDM

The expression of the characteristic polynomial (12) shows that the product of the eigenvalues is equal to 1. The stability condition can only be satisfied in two cases: either the polynomial has a double real root which absolute value is 1, or it has a pair of complex-conjugate roots lying on the unit circle. Therefore:

$$0 \geq \left(\frac{r^2}{2} - 1\right)^2 - 1 = \frac{r^2}{2}\left(\frac{r^2}{2} - 2\right). \tag{14}$$

(14) can be satisfied only if $r \leq 2$ ($r = \omega\, \Delta t$ and positive real number). Thus, the CDM is conditionally stable, and the stability limit is

$$\Delta t \leq \frac{2}{\omega} = \frac{T}{\pi}. \tag{15}$$

3.2. The amplitude decay in the case of CDM

On the stability domain:

$$\lambda_{1;2} = 1 - \frac{r^2}{2} \pm i\sqrt{1 - \left(\frac{r^2}{2} - 1\right)^2}. \tag{16}$$

Comparing expression (16) with (6), it can be seen that within the stability range, the absolute value of the eigenvalues is 1:

$$|\lambda_{1;2}| = |G| = 1, \tag{17}$$

this means the amplitude of the oscillation is preserved exactly, there is no numerical damping (AD) in the case of CDM.

3.3. The period elongation in the case of CDM

(16) can be used to determine the argument of the eigenvalues:

$$\varphi^{(CDM)} = \arccos\left(1 - \frac{r^2}{2}\right). \tag{18}$$

Entering the complex argument resulting from (16) into expressions (9) yields the relative period error:

$$p_{error}^{(CDM)} = \frac{r}{\arccos\left(1 - \frac{r^2}{2}\right)} - 1, \tag{19}$$

$$p_{error}^{(CDM)} = -\frac{r^2}{24} - \frac{17}{5760}\, r^4 + \mathcal{O}(r^5) = -\frac{r^2}{24} + \mathcal{O}(r^4). \tag{20}$$

For small r, the CDM exhibits a negative phase error. The numerical solution oscillates faster, and the period becomes shorter.

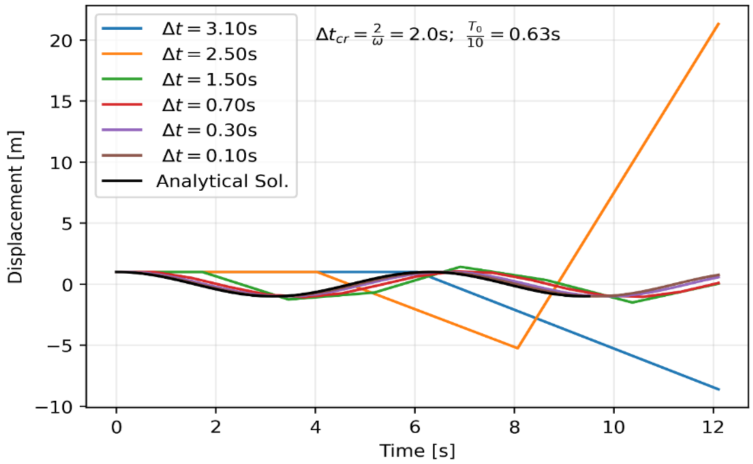

Figure 2. *Numerical solutions of the harmonic oscillator using the Central Difference Method for various time step sizes, compared to the analytical solution.*

Numerical solutions agree well with the analytical solution when small time step sizes are used (below 10% of the period), as shown in *Figure 2*, with no amplitude decay (numerical damping). For time steps larger than Δt_{cr}, the method becomes inaccurate, and the solution diverges significantly. These observations illustrate the conditional stability of the Central Difference Method.

4. The Wilson-θ Method (WθM)

The basic idea behind the Wilson–θ Method is that it assumes a linear acceleration change between times t and $t + \theta\,\Delta t$ $(0 < \theta \in \mathbb{R})$. Based on the formulas published in sources (Györgyi, 2006) and (Bathe et al., 1976), the recursive equation for undamped free vibration with one degree of freedom can be written as follows:

$$\underline{\mathbf{z}}_{n+1}^{W\theta} = \underline{\underline{\mathbf{A}}}^{W\theta}\,\underline{\mathbf{z}}_{n}^{W\theta}. \tag{21}$$

In *equation (21)* $\underline{\mathbf{z}}_{n+1}^{W\theta} = [x_{n+1};\ \dot{x}_{n+1};\ \ddot{x}_{n+1}]^T$, $\underline{\mathbf{z}}_{n}^{W\theta} = [x_n;\ \dot{x}_{n+1};\ \ddot{x}_n]^T$ and

$$\underline{\underline{\mathbf{A}}}^{W\theta} = \begin{bmatrix} \frac{6m}{\Delta t^2\omega^2\theta^5+6\theta^3}+1-\frac{1}{\theta^3} & \Delta t-\frac{\Delta t}{\theta^2}+\frac{6m}{\Delta t\omega^2\theta^4+\frac{6\theta^2}{\Delta t}} & \frac{\Delta t^2}{2}-\frac{\Delta t^2}{2\theta}+\frac{2m}{\omega^2\theta^3+\frac{6\theta}{\Delta t^2}} \\ \frac{18m}{\Delta t^3\omega^2\theta^5+6\Delta t\theta^3}-\frac{3}{\Delta t\theta^3} & \frac{18m}{\Delta t^2\omega^2\theta^4+6\theta^2}+1-\frac{3}{\theta^2} & \Delta t-\frac{3\Delta t}{2\theta}+\frac{6m}{\Delta t\omega^2\theta^3+\frac{6\theta}{\Delta t}} \\ \frac{36m}{\Delta t^4\omega^2\theta^5+6\Delta t^2\theta^3}-\frac{6}{\Delta t^2\theta^3} & \frac{36m}{\Delta t^3\omega^2\theta^4+6\Delta t\theta^2}-\frac{6}{\Delta t\theta^2} & \frac{12m}{\Delta t^2\omega^2\theta^3+6\theta}+1-\frac{3}{\theta} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{6m-r^2\theta^2+\theta^3(r^2\theta^2+6)-6}{\theta^3(r^2\theta^2+6)} & \frac{\Delta t\left(6m-r^2\theta^2+\theta^2(r^2\theta^2+6)-6\right)}{\theta^2(r^2\theta^2+6)} & \frac{\Delta t^2\left(4m-r^2\theta^2+\theta(r^2\theta^2+6)-6\right)}{2\theta(r^2\theta^2+6)} \\ \frac{3(6m-r^2\theta^2-6)}{\Delta t\theta^3(r^2\theta^2+6)} & \frac{18m}{r^2\theta^4+6\theta^2}+1-\frac{3}{\theta^2} & \frac{\Delta t\left(12m-3r^2\theta^2+2\theta(r^2\theta^2+6)-18\right)}{2\theta(r^2\theta^2+6)} \\ \frac{6(6m-r^2\theta^2-6)}{\Delta t^2\theta^3(r^2\theta^2+6)} & \frac{6(6m-r^2\theta^2-6)}{\Delta t\theta^2(r^2\theta^2+6)} & \frac{12m}{r^2\theta^3+6\theta}+1-\frac{3}{\theta} \end{bmatrix}.$$

Writing the characteristic polynomial of operator $\underline{\underline{\mathbf{A}}}^{W\theta}$ in monic form:

$$p_{W\theta}(\mu) = \mu^3 + a_1\,\mu^2 + a_2\,\mu + a_3, \tag{22}$$

$$a_1 = \frac{-12m\theta^2-18m\theta-6m-3r^2\theta^5+3r^2\theta^4+3r^2\theta^3+r^2\theta^2-18\theta^3+18\theta^2+18\theta+6}{\theta^3(r^2\theta^2+6)}, \tag{23}$$

$$a_2 = \frac{24m\theta^2-24m+3r^2\theta^5-6r^2\theta^4+4r^2\theta^2+18\theta^3-36\theta^2+24}{\theta^3(r^2\theta^2+6)}, \tag{24}$$

$$a_3 = \frac{-12m\theta^2+18m\theta-6m-r^2\theta^5+3r^2\theta^4-3r^2\theta^3+r^2\theta^2-6\theta^3+18\theta^2-18\theta+6}{\theta^3(r^2\theta^2+6)}. \tag{25}$$

4.1. The stability of WθM

θ is the factor that regulates the stability of the process. Below, we use the Jury test to determine the values of θ at which the scheme will be stable regardless of the time step and frequency magnitude. According to the Jury test, if all the following inequalities are satisfied for the coefficients of the third-degree polynomial written in the form of expression (22), then and only then are its roots located within the unit circle (Jury, 1963; Gardini, 2021):

$$1 + a_1 + a_2 + a_3 > 0, \tag{c1}$$

$$1 - a_1 + a_2 - a_3 > 0, \tag{c2}$$

$$1 - a_2 - a_3^2 + a_1\,a_3 > 0, \tag{c3}$$

$$|a_3| < 1. \tag{c4}$$

We want to obtain stability independent of frequency and time step size.
Condition (c1) does not impose a limit on θ,
Condition (c2) is satisfied if $0 < \theta < \frac{1}{2}$ or $\frac{1+\sqrt{3}}{2} < \theta$,
Condition (c3) is satisfied if $1 < \theta$, or $\theta < \frac{1}{2}$,
Condition (c4) is satisfied if $\frac{1}{2} < \theta$.

All the above conditions are satisfied if $\theta > \theta_{crit} = \frac{1+\sqrt{3}}{2} = 1.37$, in which case the process is unconditionally stable.

4.2. The amplitude decay in the case of WθM

For simplicity, let $m = 1$ kg. Instead of writing down analytical solutions, I will attempt to expand the real solution around 0 in terms of r. For $m = 1$ kg and $r = 0$, the polynomial (22) takes the following form:

$$p_{W\theta}(\mu) = \frac{(\mu-1)^2\,(\mu\,\theta-\theta+1)}{\theta}, \tag{26}$$

whose real root is: $\mu_{real0} = 1 - \frac{1}{\theta}$. The coefficients of the power series of the real root can be determined from the condition that it gives the zero of the characteristic polynomials:

$$\mu_{real}(r) = \left(\frac{7\theta^3}{18} - \frac{11\theta^2}{12} + \frac{25\theta}{36} - \frac{1}{6}\right) r^4 + \frac{(-2\theta^2+3\theta-1)}{6\theta} r^2 + \frac{\theta-1}{\theta} + \mathcal{O}(r^5) \tag{27}$$

From Vieta's relation we obtain the modulus of the complex eigenvalues as:

$$|G| = \sqrt{-\frac{a_3}{\mu_{real}}}, \tag{28}$$

A Taylor expansion in the non-dimensional time step r yields:

$$|G(r,\theta)| = 1 + \frac{\theta\,(-2\theta^2+3\,\theta-1)}{12} r^4 + \mathcal{O}(r^5), \tag{29}$$

In the unconditionally stable range ($\theta > \frac{1+\sqrt{3}}{2} = 1.37$), the coefficient in brackets is negative, hence $|G| < 1$, the WθM introduces numerical damping which is of fourth order in the time step. (Since the amplitude is multiplied by $|G|$ at each time step, the numerical damping per step is proportional to $1 - |G|$.)

4.3. The period elongation in the case of WθM

Retaining the previous assumptions and notations, based on Vieta's formulas, the sum of the roots is (λ and $\bar{\lambda}$ being the complex conjugate pair of roots):

$$\lambda + \bar{\lambda} + \mu_{real} = -a_1 = \frac{-3r^2\theta^3+3r^2\theta^2+3r^2\theta+r^2-18\theta+6}{\theta(r^2\theta^2+6)}, \tag{30}$$

$$\lambda + \bar{\lambda} = 2\,|G|\,\cos\varphi. \tag{31}$$

Using formulas (30) and (31) and then applying series expansion, we obtain the following expression:

$$p_{error}^{(W\theta M)} = \left(-\frac{\theta^2}{4} + \frac{\theta}{4} - \frac{1}{24}\right)^2 r^4 + \left(\frac{\theta^2}{4} - \frac{\theta}{4} + \frac{1}{24}\right) r^2 + \mathcal{O}(r^5) = \left(\frac{\theta^2}{4} - \frac{\theta}{4} + \frac{1}{24}\right) r^2 + \mathcal{O}(r^4). \tag{32}$$

Thus the WθM exhibits a second-order phase error, which is positive in the unconditionally stable range, indicating a period elongation (phase lag). In summary, Vieta's relations allow the numerical damping and dispersion of the WθM to be obtained entirely from the real eigenvalue and the polynomial coefficients, without computing the complex roots explicitly.

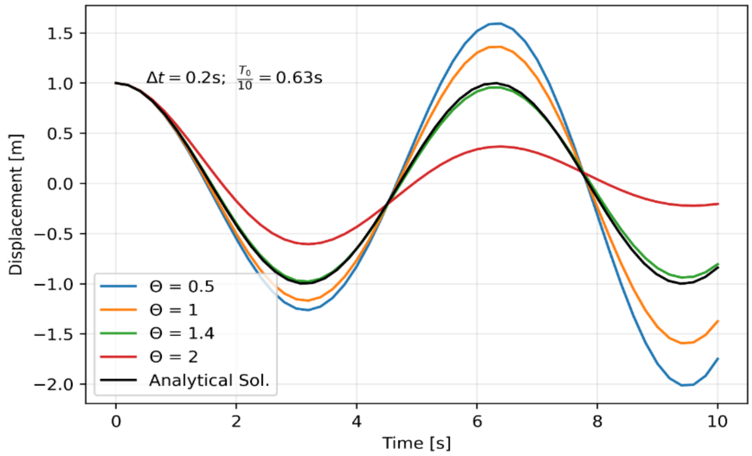

Figure 3. *Numerical solutions of the harmonic oscillator obtained using the Wilson–θ Method for different θ values, compared with the analytical solution.*

Numerical solutions are unstable for $\theta < 1.4$ *(Figure 3)*. At $\theta = 1.4$, the response is stable and remains close to the analytical solution. For larger θ values, amplitude decay and period elongation increase, resulting in a growing deviation from the analytical response.

5. Summary

This study examined the stability and accuracy of the Central Difference Method and the Wilson–θ scheme within a unified characteristic-polynomial framework. Stability was assessed using the Jury criterion, while numerical damping (AD) and period elongation (PE) were obtained from the modulus and argument of the dominant eigenvalues. It was shown that CDM is conditionally stable but amplitude-preserving in its stable range, whereas the Wilson–θ Method becomes unconditionally stable for $\theta > 1.37$, at the price of fourth-order numerical damping and second-order phase error. These results illustrate how even widely used classical schemes introduce artificial dissipation and phase lag in long-time simulations, motivating the development of advanced time-integrating *hp*-FEM formulations.

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