THERMOELASTIC PROBLEMS OF FUNCTIONALLY GRADED MULTILAYERED INCOMPRESSIBLE SPHERICAL BODIES

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Abstract
The main objective of this paper is to determine the stresses and displacements in radially graded multilayered spherical bodies made from incompressible functionally graded materials. The material properties are arbitrary functions of the radial coordinate. The body is subjected to constant pressure and a temperature field, which is an arbitrary function of the radial coordinate. An analytical method is presented to tackle these problems then compared to solutions coming from finite element simulations.

Keywords: *FGM, incompressible, sphere, multilayered*

1. Introduction

In recent decades layered composite and functionally graded materials (FGMs) have been widely used in numerous engineering applications due to their excellent material behavior. These materials have improved thermal resistance and mechanical properties. The smooth transition between the components decreases the chance of cracking and debonding under thermal and mechanical loads. A lot of papers and books deal with the mechanics of these advanced materials. Books by Boley and Weiner [1], Lekhnitskii [2] and Lomakin [3] give solutions to many linearly elastic problem for non-homogeneous bodies, furthermore there are lots of papers, such as [4-11] that tackle the thermomechanical problems of different simple functionally graded and multilayered components (e.g. beams or disks) from different aspects. Several papers can be found in the topic of functionally graded spherical bodies. Lutz and Zimmerman in [12] and [13] give analytical solutions for the stress and displacement fields within functionally graded spherical bodies and circular cylinders. Papers [14] and [15] present analytical solutions to obtain the radial, tangential and effective stresses within thick spherical pressure vessels made of FGMs subjected to axisymmetric simple thermomechanical loads. The material properties are assumed to be graded in the radial direction based on the power-law function of the radial coordinate but the Poisson ratio has constant value. Paper [16] gives solutions for thick radially graded cylinders and spherical pressure vessels where the material properties of FGMs is assumed to be exponential function form. In paper [17] a general thermoelastic analysis of one-dimensional steady-state thermal stresses in a hollow thick cylinder made of functionally graded material is presented. Kar and Kanoria [18] dealt with the determination of thermo-elastic interaction due to a step input of temperature on the boundaries of a radially graded orthotropic hollow sphere in the context of linear theories of generalized thermo-elasticity using a vector-matrix formalism and eigenvalue solution approach. Bich and Tung [19] presented an analytical approach to investigate the non-linear axisymmetric response of functionally graded shallow spherical shells subjected to constant external pressure incorporating the effects of temperature field and imperfections. Gönczi [20] derived an analytical method to calculate the stresses in multilayered spheres, then used this method to tackle a general problem of temperature dependent radially graded spherical pressure vessels. Ye et. al [21] investigated the free vibration of laminated functionally graded spherical shells with general boundary conditions and arbitrary geometric parameters with power law distribution based on three-dimension linearized shell theory and Rayleigh-Ritz method. Paper [22] studied the elastic – perfectly plastic thermal stress analysis of radially graded spheres and the material properties are power functions of the radial coordinate. Nematolli et. al. [23] presented an analytical solution of thermal and mechanical displacements and stresses for a thick-walled rotating spherical pressure vessel made of functionally graded materials in a uniform magnetic field assuming power law distribution along the thickness. In paper [24] a numerical algorithm is developed to describe the thermoelastic wave in multilayered spherical shells with functionally graded layers under thermal boundary conditions based on Lord-Shulman generalized coupled thermoelasticity theory. Viola et. al. [25] studied the static behavior of functionally graded spherical shells and panels subjected to uniform loadings, the material properties are graded in the thickness direction according to a four- parameter power law, and a *GDQ* numerical technique is used to solve the system of differential equations. Arefi and Zenkour [26] investigated the problems of functionally graded spherical pressure vessels using non-linear shell theory and Adomians decomposition method. In paper [27] a closed form solution was developed for spheres, in which the functions of the material properties – except the Poisson's ratio - are power law functions. Then a multilayered approach was used with Boussinesq displacement potentials to deal with the optimization problem of hollow spheres. In [28] a closed form analytical solution is presented for special thermoelastoplastic problems of thick-walled spheres. Akinlabi et. al. [29] developed a thermoelastopalstic method to calculate the stresses and displacements in functionally graded spheres after thermal treatment. In [30] an analytical method is presented to calculate the thermal stresses in incompressible radially graded spheres. Paper [31] uses perturbation technique and solves the shell problem in case of temperature dependent material properties.

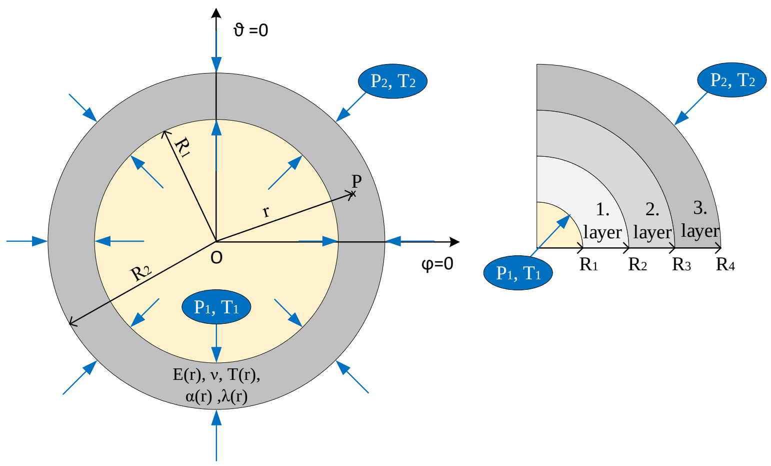

***Figure 1**. The sketch of the problem.*

A multilayered incompressible radially graded hollow sphere is considered. The sketch of the problem can be seen in Fig. 1. The number of layers is denoted by N, the material properties are arbitrary functions of the radial coordinate. Spherical coordinate system will be used. Our aim is to determine the analytical solution for the stress and displacement fields within the radially graded multilayered sphere.

2. Governing equations

The Poisson's ratio of incompressible materials is $\nu = 0.5$ thus the Young modulus and the shear modulus have the relation $E = 3G$. This problem is an axisymmetric problem where the displacement field is $\mathbf{u} = u(r)\mathbf{e}_r$ and the kinematic equations are [1-3, 32]

$$\varepsilon_r(r) = \frac{\mathrm{d}u(r)}{\mathrm{d}r}, \quad \varepsilon_\varphi(r) = \varepsilon_\vartheta(r) = \frac{u(r)}{r}. \tag{1}$$

The normal strains can be calculated as the sum of its mechanical and thermal components, furthermore the constitutive law can be expressed as

$$\varepsilon_r(r) = \frac{\sigma_r(r) - \sigma_\varphi(r)}{E(r)} + \alpha(r)T(r), \tag{2}$$

$$\varepsilon_\varphi(r) = \frac{-\sigma_r(r) + \sigma_\varphi(r)}{2E(r)} + \alpha(r)T(r), \tag{3}$$

where $T(r)$ denotes the temperature field, σ_r, σ_φ, σ_ϑ are the radial and hoop (or tangential normal) stresses, ε_r, ε_φ, ε_ϑ denote the normal strain coordinates of the strain tensor. The trace of the strain tensor yields to

$$\varepsilon = \varepsilon_r + \varepsilon_\varphi + \varepsilon_\vartheta = \varepsilon_r + 2\varepsilon_\varphi = 3\alpha T\,. \tag{4}$$

The combination of Eqs. (1-4) leads to the following differential equations

$$\varepsilon(r) = \frac{du(r)}{dr} + 2\frac{u(r)}{r} = \frac{1}{r^2}\frac{d}{dr}\left(ur^2\right). \tag{5}$$

The solution for the radial displacement coordinate can be given as

$$u(r) = \frac{3\int\limits_{R_1}^{r} \rho^2 \alpha(\rho) T(\rho) d\rho}{r^2} + \frac{A_1}{r^2}. \tag{6}$$

The previous expression can be simplified by the following notation

$$V_1(r) = \int\limits_{R_1}^{r} \rho^2 \alpha(\rho) T(\rho) d\rho\,,\ u(r) = 3\frac{V_1(r)}{r^2} + \frac{A_1}{r^2}\,. \tag{7}$$

In spherical bodies the equilibrium equation has the following form

$$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\varphi}{r} + b_r = 0\,, \tag{8}$$

where b_r denotes the radial body force. The stresses in incompressible materials can be expressed as [1, 32]

$$\sigma_r = 2G\varepsilon_r^m + \sigma_0 = \frac{2}{3}E\varepsilon_r^m + \sigma_0, \tag{9}$$

$$\sigma_\varphi = \sigma_\vartheta = 2G\varepsilon_\varphi^m + \sigma_0 = \frac{2}{3}E\varepsilon_\varphi^m + \sigma_0, \tag{10}$$

where

$$\sigma_0 = \frac{\sigma_r + \sigma_\varphi + \sigma_\vartheta}{3} = \frac{\sigma_r + 2\sigma_\varphi}{3}. \tag{11}$$

The combination of eqs. (8-11) leads to

$$\frac{d\sigma_r}{dr} = 4A_1\frac{E(r)}{r^4} + 12\frac{E(r)F_1(r)}{r^4} - 4\frac{E(r)\alpha(r)T(r)}{r} - b_r(r), \tag{12}$$

$$\sigma_r(r) = 4C_1\int_{R_1}^{r}\frac{E(\rho)}{\rho^4}\mathrm{d}\rho + 12\int_{R_1}^{r}\frac{E(\rho)F_1(\rho)}{\rho^4}\mathrm{d}\rho - 4\int_{R_1}^{r}\frac{E(\rho)\alpha(\rho)T(\rho)}{\rho}\mathrm{d}\rho - \int_{R_1}^{r}b_r(\rho)\mathrm{d}\rho + A_2. \tag{13}$$

The unknown constants of integrations can be calculated from the boundary conditions. When the pressures at the inner and outer boundary surfaces are given we have

$$p_1 = -\sigma_r(R_1),\;\; p_2 = -\sigma_r(R_2)\,. \tag{14}$$

When mixed boundary conditions with the radial displacement of the outer boundary is given (which means that there is a rigid casing around the spherical body):

$$u_{presc} = u(R_2),\;\; p_1 = -\sigma_r(R_1)\,. \tag{15}$$

The tangential normal stresses can be expressed as the combination of the previous equations:

$$\sigma_\varphi(r) = \frac{1}{2}r\left(\frac{d\sigma_r}{dr} + b_r\right) + \sigma_r. \tag{16}$$

3. Numerical examples

In the numerical example the spherical body is divided into 3 radially graded layers (where the thickness of every layer is 0.03 m). Each layer has different material properties, furthermore the temperature field is given. These properties and fields are described with the following functions:

$$\alpha_1=\alpha_{01}\left(\frac{r}{R_1}\right)^{m_1},\ \alpha_2=\alpha_{02}e^{rm},\ \alpha_3=\alpha_{03}\sin(m_4r^2), \tag{17}$$

$$E_1=E_{01}\left(\frac{r}{R_1}\right)^{m_1},\ E_2=E_0e^{rm_3},\ E_3=E_{03}\sin(m_4r^2), \tag{18}$$

$$T(r)=-550+\frac{200}{r^2}. \tag{19}$$

The following data will be used for the computations:

$$R_1=0.5\text{m}, R_2=0.59\text{m}, \alpha_{01}=1.2\cdot10^{-6}\frac{1}{\text{K}}, \alpha_{02}=3.6\cdot10^{-6}\frac{1}{\text{K}}, \alpha_{03}=1.2\cdot10^{-6}\frac{1}{\text{K}}, E_{01}=210\text{GPa};$$
$$E_{02}=600\text{GPa}, E_{03}=210\text{GPa}, m_1=2, m_3=-2.1, m_4=1.5, p_1=50\text{MPa}, p_2=5\text{MPa}.$$

Figures 2, 3 and 4 depict comparisons of the results from the analytical (calculated in MAPLE) and finite element simulations (in Abaqus, [33]). In the finite element simulations, the functionally graded sphere is modelled as a multilayer body with N=9 and N=18 homogeneous spherical layers. Table 1 shows the exact results of each calculation at specified distances and the relative errors.

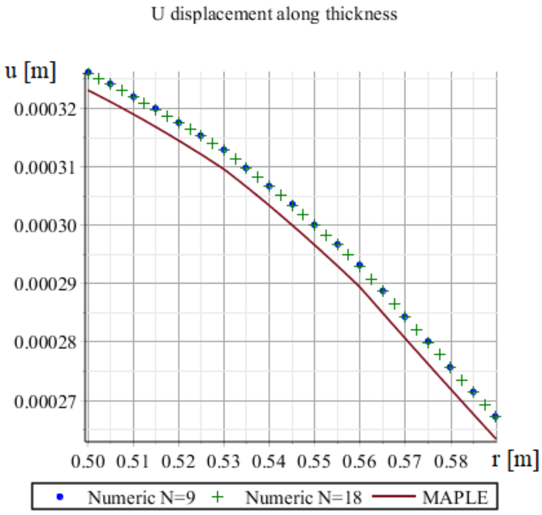

***Figure 2**. Comparison of the displacements.*

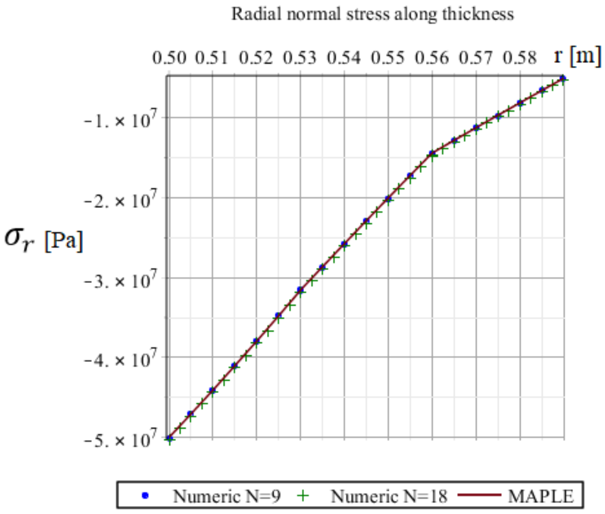

***Figure 3**. Comparison of the radial stresses.*

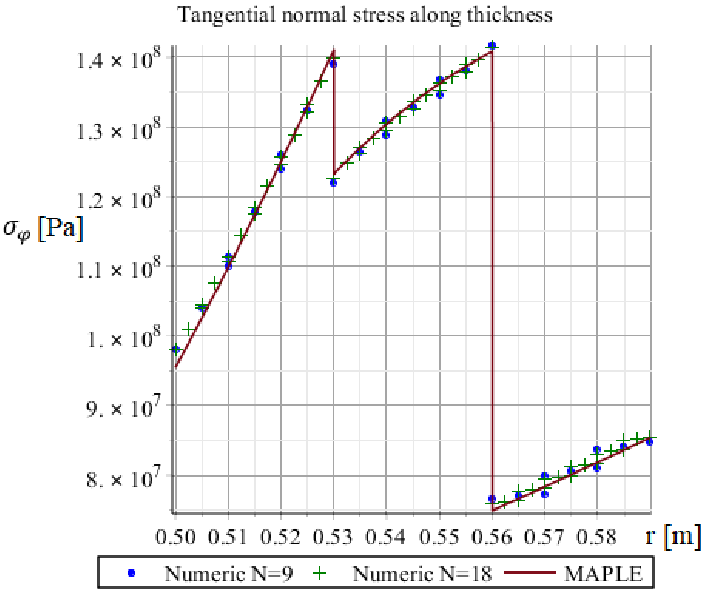

***Figure 4**. Comparison of the tangential stresses.*

***Table 1**. Exact values and relative errors.*

Displacement (m)	r=0.5	r= 0.52	r=0.54	r=0.56	r=0.58	r=0.59
MAPLE	3,23E-04	3,15E-04	3,03E-04	2,89E-04	2,72E-04	2,63E-04
Numeric N=18	3,26E-04	3,18E-04	3,07E-04	2,93E-04	2,76E-04	2,67E-04
Numeric N=9	3,26E-04	3,18E-04	3,07E-04	2,93E-04	2,76E-04	2,67E-04
Relative error N=18 (%)	-0,91	-0,97	-1,10	-1,26	-1,40	-1,46
Relative error N=9 (%)	-0,93	-0,99	-1,12	-1,28	-1,41	-1,47

Radial normal stress (Pa)	r=0.5	r= 0.52	r=0.54	r=0.56	r=0.58	r=0.59
MAPLE	-5,00E+07	-3,79E+07	-2,57E+07	-1,44E+07	-8,08E+06	-5,00E+06
Numeric N=18	-5,03E+07	-3,82E+07	-2,60E+07	-1,46E+07	-8,29E+06	-5,19E+06
Numeric N=9	-5,01E+07	-3,80E+07	-2,58E+07	-1,44E+07	-8,13E+06	-5,03E+06
Relative error N=18 (%)	-0,62	-0,75	-1,22	-1,55	-2,51	-3,73
Relative error N=9 (%)	-0,17	-0,09	-0,26	-0,49	-0,61	-0,67

Tangential normal stress (Pa)	r=0.5	r= 0.52	r=0.54	r=0.56	r=0.58	r=0.59
MAPLE	9,54E+07	1,25E+08	1,30E+08	7,49E+07	8,19E+07	8,54E+07
Numeric N=18	9,80E+07	1,26E+08	1,30E+08	7,60E+07	8,31E+07	8,55E+07
Numeric N=9	9,80E+07	1,26E+08	1,29E+08	7,65E+07	8,36E+07	8,47E+07
Relative error N=18 (%)	-2,70	-0,48	0,67	-1,39	-1,51	-0,07
Relative error N=9 (%)	-2,64	-0,66	1,22	-2,09	-2,16	0,87

4. Summary

An analytical method was presented to tackle the thermoelastic problem of functionally graded incompressible hollow spheres subjected to combined mechanical and thermal loads. The material properties and the temperature field were arbitrary functions of the radial coordinate. The stress and displacement fields were calculated then compared to results coming from finite element simulations. The results are in good agreement.

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