

SOME REMARKS OF RANDOM GRAPHS

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Abstract

At the beginning of the paper, the known models are briefly presented (The Barabási–Albert model, Watts–Strogatz model, Erdős–Rényi model). In the later part of the paper, some results are presented, which are valid in the Erdős–Rényi model and are also related to the dominant sets of graphs. These results will be considered further in a later paper (Bacsó et al., 2021).

Keywords: Erdős–Rényi model, random graphs, dominant sets

1. Introduction

Several random graph models exist. Let us first review the main random graph theory models.

Today's intensively researched model is the Barabási–Albert model: The Barabási–Albert model is an algorithm for generating random Scalefree networks using a preferential attachment mechanism. Several natural and human-made systems, including the Internet, the world wide web, citation networks, and some social networks are thought to be approximately scale-free and certainly contain few nodes (called hubs) with unusually high degree as compared to the other nodes of the network. The Barabási–Albert model tries to explain the existence of such nodes in real networks. The algorithm is named for its inventors Albert–László Barabási and Réka Albert.

Preferential attachment means that the more connected a node is, the more likely it is to receive new links. Nodes with a higher degree have a stronger ability to grab links added to the network. Intuitively, the preferential attachment can be understood if we think in terms of social networks connecting people. Here a link from A to B means that person A "knows" or "is acquainted with" person B. Heavily linked nodes represent well-known people with lots of relations. When a newcomer enters the community, they are more likely to become acquainted with one of those more visible people rather than with a relative unknown. The The Barabási–Albert model was proposed by assuming that in the World Wide Web, new pages link preferentially to hubs, i.e., very well-known sites such as Google, rather than to pages that hardly anyone knows. If someone selects a new page to link to by randomly choosing an existing link, the probability of selecting a particular page would be proportional to its degree.

The Barabási–Albert is a scale-free network: the scale-free network is a network whose degree distribution follows a power law, at least asymptotically. That is, the fraction $P(k)$ of nodes in the network having k connections to other nodes goes for large values of k as

$$P(k) \sim k^{-\gamma},$$

where γ is a parameter whose value is typically in the range $2 < \gamma < 3$ (wherein the second moment (scale parameter) $k^{-\gamma}$ is infinite but the first moment is finite), although occasionally it may lie outside these bounds.

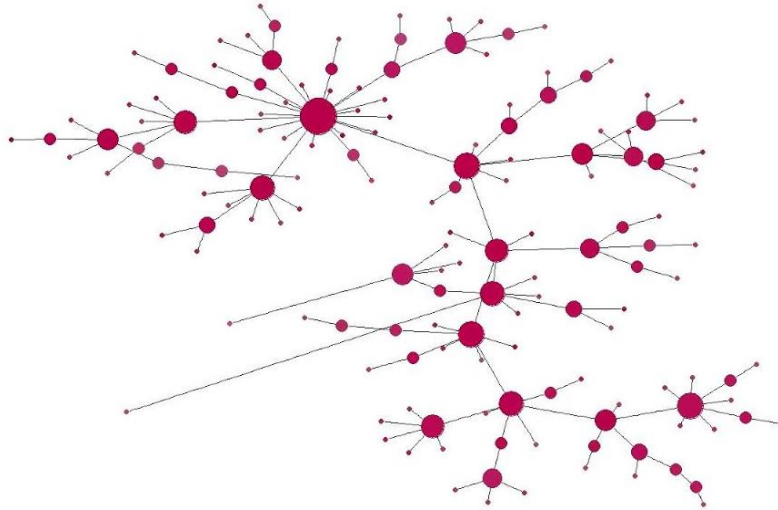


Figure 1. *The Barabási-Albert model*

The Watts–Strogatz model is a random graph generation model that produces graphs with small-world properties, including short average path lengths and high clustering. It was proposed by Duncan J. Watts and Steven Strogatz in their article published in 1998 in the Nature scientific journal (Watts and Strogatz, 1998).

Watts-Strogatz model $N=20$, $K=4$, $\beta=0.2$

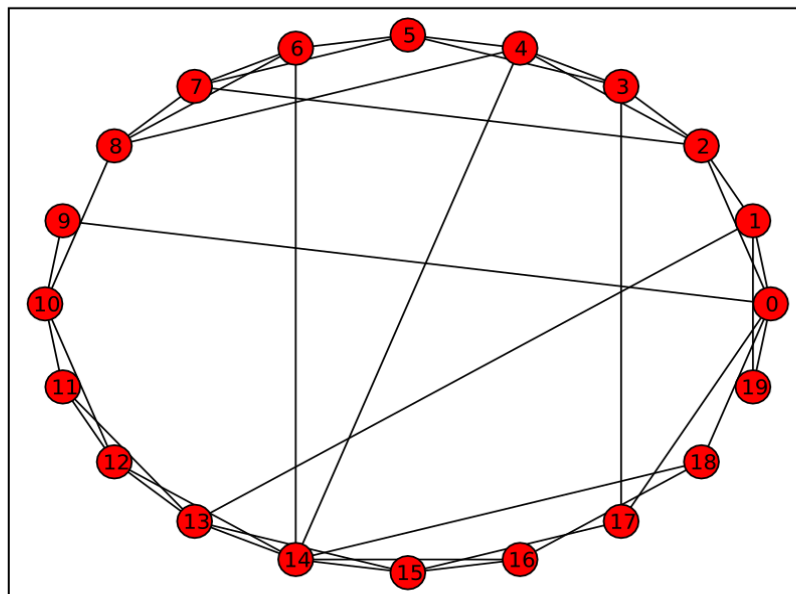


Figure 2. *The Watts-Strogatz model*

We investigate a classical standard random graph model. This model due to Erdős and Rényi: here every possible edge is created with same p probability: each edge has a fixed probability of being

present or absent, independently of the other edges. In other work, the probability p is not constant (Bacsó et al., 2021).

Let us consider the standard random graph model, namely the probability that a pair of vertices (points) is an edge, will be p and the pairs are completely independent. We take a random graph G on the vertices $1, 2, \dots, n$.

In the $G(n; p)$ model, a graph is constructed by connecting labelled nodes randomly. Each edge is included in the graph with probability p , independently from every other edge.

With the notation above, a graph in $G(n; p)$ has on average $\binom{n}{2} p$ edges.

The distribution of the degree of any particular vertex is binomial:

$$P(\deg(v) = k) = \binom{n-1}{k} p^k (1-p)^{n-k-1},$$

where n is the total number of vertices in the graph.

Since

$$P(\deg(v) = k) = \frac{(np)^k e^{-np}}{k!},$$

as $n \rightarrow \infty$ and $np = \text{constant}$ this distribution is Poisson for large n .

If $np > 1$, then a graph in $G(n; p)$ will almost surely have no connected components of size larger than $O(\log(n))$.

If $np = 1$, then a graph in $G(n; p)$ will almost surely have largest component whose size order $n^{2/3}$.

If $np \rightarrow c > 1$, then a graph in $G(n; p)$ will almost surely have a unique giant component containing a positive fraction of the vertices. No other component will contain more than $O(\log(n))$ vertices.

If $p < \frac{(1-\varepsilon)\ln n}{n}$, then a graph in $G(n; p)$ will almost surely contain isolated vertices and thus be disconnected.

If $p > \frac{(1-\varepsilon)\ln n}{n}$, then a graph in $G(n; p)$ will almost surely be connected.

This $\frac{\ln(n)}{n}$ is a sharp threshold function for the connectedness of $G(n; p)$.

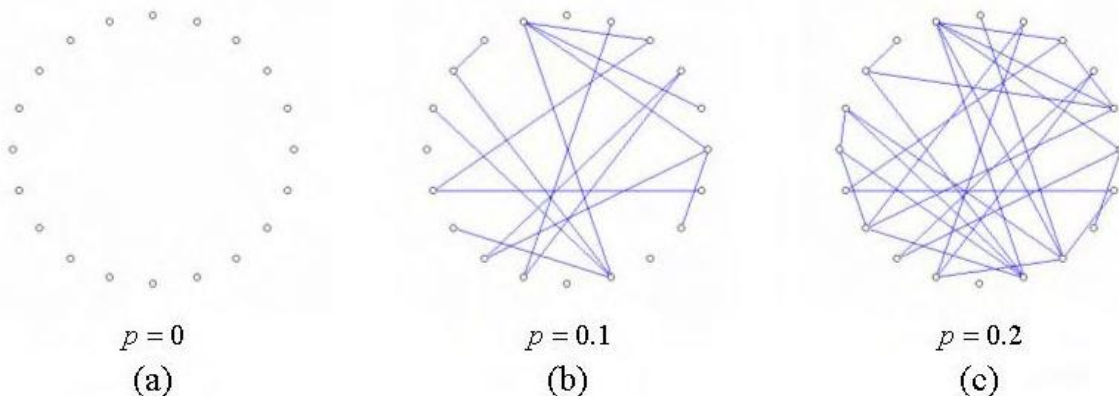


Figure 3. Graphs generated by Erdős and Rényi model ($p=0, p=0.1, p=0.2$)

2. Results and proofs

We investigate the Erdős-Rényi model but we suppose that the graph is connected and the graph has a dominating set (a dominating set for a graph $G = (V;E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D).

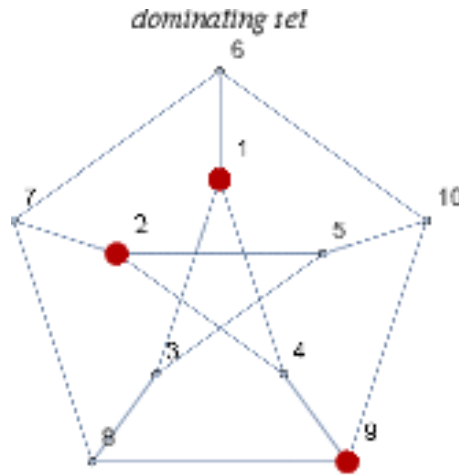


Figure 4. A dominating set in graph (the points of the dominant set are marked in red)

In this model, given two points x and z , for a third point y , the probability that y is the common neighbour of them, is p^2 , because of independence. If y runs through all the points different from x and z , the probability that none of them is a common neighbour of the two fixed points is $(1 - p^2)^{n-2}$, from the independence again. Consequently,

Theorem 1. For all $x \neq z$, the probability that the distance of x and z is at most 2, is at least $P(n) := 1 - (1 - p^2)^{n-2}$. ($P(n)$ tends to 1, of course).

Theorem 2. 'At least' is written here because x and z may be adjacent.

First, ordinary domination is considered.

As we know, the probability that the set $D := \{1, 2, \dots, r\}$ is dominating in G tends to 1 if $r = \log_q n + \epsilon$ where $q = 1/p$ and ϵ tends to infinity.

Now we add auxiliary edges such that they yield a (n arbitrary) tree T on D . Let us partition the vertex set $V - D$ into $r - 1$ (essentially) equal-size classes in such a way that each class C_i corresponds to some auxiliary edge e_i of T between x_i and z_i ($i = 1, 2, \dots, r-1$). The size of one class is about $(n-r)/(r-1)$, which is asymptotic to n/r . By Theorem 1, if we pick a pair (x_i, z_i) , in the subgraph induced by $M_i := C_i \cup \{x_i, z_i\}$ the probability of the event A_i that their distance is at most 2, is at least $s_i := 1 - (1 - p^2)^{n_i}$ where $n_i := |C_i|$.

If for every i , the event A_i occurs, we may construct a (generally larger) dominating set of G , namely, we add a common neighbour y_i of x_i and z_i in the subgraph $G | M_i$ (or they are adjacent).

Note that, because of the edge-disjointness of the subgraphs induced by the vertex sets M_i , the events A_i are independent. Consequently,

Theorem 3. The probability that, in such a way, we obtain a connected dominating subgraph, is at least $\prod_{i=1}^{r-1} s_i$.

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